JOURNAL OF MATHEMATICS AND PHYSICS

Massachusetts Institute of Technology

Volume V
1925 – 1926

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THEORY OF THE CONTINUOUS X-RAY SPECTRUM*

By M. Sandoval Vallarta

A satisfactory theory of the continuous X-ray spectrum is, to the writer's knowledge, still lacking. Classical electrodynamics leads to conclusions in violent clash with experiment; in fact the former would lead to expect a continuous spectrum having all frequencies,[1] whereas the latter shows definitely the existence of a sharp maximum frequency limit. In the usual quantum interpretation of this phenomenon, deduced from the Einstein-Planck law $h\nu = eV$, the limit wave-length depends only on the voltage applied to the tube and varies inversely with it. This is in accordance with experiments performed to date. As far as the writer is aware, however, no quantum theory of the continuous X-ray spectrum has been offered, although several suggestions have been put forward.[2]

In the present paper we intend to advance an alternative quantum theory of the continuous X-ray spectrum, which makes use of the following principles and is based on the following assumptions:

(1) The continuous spectrum is due *solely* to the stoppage of electrons on the anticathode. These electrons, emitted from the cathode or otherwise, are assumed to be endowed with a definite kinetic energy which is equal to the work done on them by the electric field within the tube. A first consequence of these assumptions is that the continuous and the characteristic X-ray spectra have different mechanisms of emission — the former by decelerated electrons, the latter by atoms excited by electron impact. Another consequence is that, at relatively large distances from the cathode, all electrons have the same kinetic energy.

(2) Radiation from an electron, due to loss of all or part of its kinetic energy, takes place in accordance with Bohr's frequency relation.

*Presented at the American Physical Society, Chicago, November 27, 1925.

[1] Max Abraham "Theorie der Elektrizitat" Vol. 2, p. 93 ff. Leipzig, 1920, also, A. Sommerfeld "Atombau and Spektrallinien," p. 32, Braunschweig, 1924.

[2] Sommerfeld, l. c. p. 748. H. A. Kramers, Philosophical Magazine, Vol. 46, p. 836, 1923; G. Wentzel, Zeitschrift für Physik, Vol. 27, p. 257, 1923.

(3) Conservation of energy and momentum hold for the system consisting of electron, anticathode atom and light-quantum.

(4) When impinging on an anticathode atom, an electron may or may not lose all its kinetic energy: the loss of energy is at random for different electrons.

(5) The law of variation of mass with velocity $m=m_0(1-\beta^2)^{-1/2}$ and its consequences hold for both atoms and electrons.

(6) The law of inertia of energy and its consequences hold for the emitted light quantum.

Further, it is assumed that there is no variation in the potential energy of either electrons or atoms, due to external or intrinsic fields of force, before and after collision.

Let now T_1 be the kinetic energy of the moving electron before impact, T_1' its energy after impact, T_2 the kinetic energy of the anticathode atom after impact, h the Planck constant, ν the frequency of the light-quantum. Then the conservation of energy together with Bohr's frequency relation give

$$T_1-(T_1'+T_2)=h\nu \tag{1}$$

but if e is the electronic charge and V the applied tube voltage, we have, according to the assumptions under (1), above,

$$T_1=eV=\text{const.} \tag{2}$$

Now, the amount of energy transferred by inelastic collision from the electron to the atom may be anything we please, in accordance with assumption (4). Therefore, the sum $T_1'+T_2$ may also have any value different from zero, since T_2 cannot be zero. For if T_2 vanished, then the only possible physical interpretation would be that there had been no collision between electron and atom, hence that no radiation had taken place. We now show that if $T_1'+T_2$ can have any value, then ν can also have any value below a certain maximum ν_0.

Suppose that we number the electrons in the cathode beam and let the kinetic energy of electron number 1 become T_{11}' after collision, that of the nth electron become T_{1n}', etc. Further let the kinetic energy of the atom in collision with electron number 1 be T_{21}, after impact, that of the atom in collision with the nth electron be T_{2n} also after impact, etc. Note that the possibility that the same atom may enter into collision with two or more

electrons is taken into account in the above formulation. Finally let $\nu_1, \nu_2, \ldots \nu_n$ etc., be the frequencies of the light-quanta due to the corresponding impacts. Then, by Equation (1)

$$T_1-(T_{11}'+T_{21})=h\nu_1$$

. . . . . .

$$T_1-(T_{1n}'+T_{2n})=h\nu_n$$

. . . . . .

and for the difference between the frequencies of the quanta emitted by impact from the nth electron and the $(n-1)$st electron, we have

$$-h(\nu_n-\nu_{n-1})=(T_{1n}'+T_{2n})-(T_{1(n-1)}'+T_{2(n-1)})$$

and since the difference between the two right-hand terms may be as small as desired, therefore the difference between the frequencies of the quanta radiated as a result of two independent elementary processes such as have been described can be made as small as desired. It follows that a set of this type of elementary processes, discontinuous in themselves but taking place at random give rise to a continuous spectrum.

The maximum frequency is obtained when $T_1'=0$, *i.e.*, when the electron is brought to a full stop by impact. This case then corresponds to the limit wave-length. We have, for this case,

$$T_1-T_2=h\nu_0 \qquad (3)$$

and we arrive thus at a simple physical interpretation of the meaning of the limit wave-length. It is seen at once that the Einstein-Planck equation $T_1=h\nu_0$ can be more nearly exact only if T_2 is negligible, *i.e.*, if the velocity imparted to the atom by impact is very small, or else if its mass is very small. We now investigate the following question: How does the limit wave-length depend on the anticathode material? As conservation of momentum holds during impact, according to our assumptions, the limit wave-length may also be expected to depend upon the direction of observation relatively to the direction of travel of the electrons.

Let m_0 be the rest-mass of the electron, β_1 the ratio of its velocity before collision to the velocity of light; c the velocity of light.

If the mass of the electron at the velocity $\beta_1 c$ is

$$m = m_0(1-\beta_1^2)^{-1/2}$$

its kinetic energy at that velocity is

$$T_1 = m_0c^2[(1-\beta_1^2)^{-1/2}-1]). \tag{4}$$

Likewise, let M_0 be the rest-mass of the anticathode atom, β_2 the ratio of its velocity after collision to the velocity of light. Its kinetic energy is, likewise

$$T_2 = M_0c^2[(1-\beta_2^2)^{-1/2}-1]. \tag{5}$$

Since $T_1 = eV$, Equation (3) gives

$$eV - M_0c^2[(1-\beta_2^2)^{-1/2}-1] = h\nu_0. \tag{6}$$

We still have to write equations expressing that momentum is conserved for the system under consideration. Now if, according to our assumptions, the mass of the light-quantum of energy $h\nu$

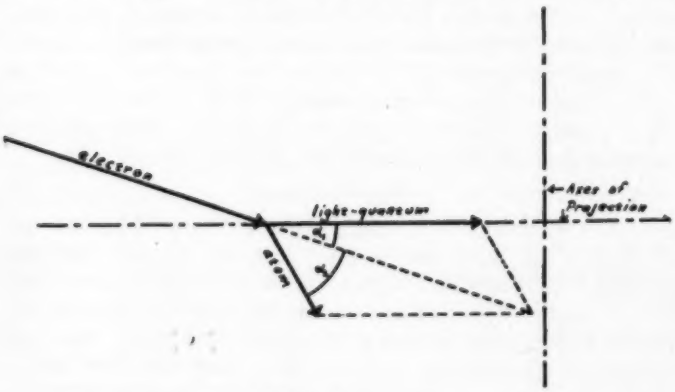


FIG. 1.

is $h\nu/c^2$ its momentum is $h\nu/c$. Projecting the three momentum vectors on an axis along the line of motion of the light-quantum (see Fig. (1)), we obtain

$$m_0\beta_1 c(1-\beta_1^2)^{-1/2}\cos\alpha_1 - h\nu_0/c = M_0\beta_2 c(1-\beta_2^2)^{-1/2}\cos(\alpha_1+\alpha_2). \tag{7}$$

Projecting now on a line perpendicular to the axis defined above, we have

$$m_0\beta_1 c(1-\beta_1^2)^{-\frac{1}{2}}\sin\alpha_1 = M_0\beta_2 c(1-\beta_2^2)^{-\frac{1}{2}}\sin(\alpha_1+\alpha_2) \quad (8)$$

α_1 and α_2 have the meanings shown in Fig. (1). Of these angles, the only one which interests us is α_1, for nothing is known concerning α_2. Accordingly, in order to obtain our final expression for ν_0, the limit frequency, in terms of tube voltage, angle of observation, mass of the anticathode atom and universal constants, we eliminate from Equations (6), (7) and (8) the parameters α_1, β_1 and β_2. From Equations (2) and (4)

$$\beta_1^2 = 1-\frac{1}{\left(\frac{eV}{m_0c^2}+1\right)^2} \quad (9)$$

and from Equation (6)

$$\beta_2^2 = 1-\frac{1}{\left(\frac{eV-h\nu_0}{M_0c^2}+1\right)^2}. \quad (10)$$

Squaring and adding Equations (7) and (8) we obtain,

$$\frac{m_0^2\beta_1^2c^2}{1-\beta_1^2}+\frac{h^2\nu_0^2}{c^2}-\frac{2m_0\beta_1 ch\nu_0}{\sqrt{1-\beta_1^2}}\cos\alpha_1 = \frac{M_0^2\beta_2^2c^2}{1-\beta_2^2} \quad (11)$$

which combined with Equations (9) and (10) gives

$$m_0eV-h\nu_0\cos\alpha_1\sqrt{\frac{e^2V^2}{c^2}+2m_0eV} = eVM_0-h\nu_0M_0-\frac{eVh\nu_0}{c^2} \quad (12)$$

This equation determines the limit frequency ν_0

$$\nu_0 = \frac{1}{h}\,\frac{eV(M_0-m_0)}{eV/c^2+M_0-\cos\alpha_1\sqrt{e^2V^2/c^2+2m_0eV}} \quad (13)$$

or the limit wave-length λ_0

$$\lambda_0 = \frac{h(eV/c^2+M_0-\cos\alpha_1\sqrt{e^2V^2/c^2+2m_0eV}}{eV(M_0-m_0)}. \quad (14)$$

It follows that the limit frequency, or the limit wave-length, depends not only on the voltage, but also on the direction of observation, *i.e.*, on the angle α_1 between the direction of motion of

the electrons and the direction along which the observation is made, and on the anticathode material.

Suppose now that two observations are made on the same tube, but in two different directions. We should expect then a *shift* of the sharp limit of the continuous spectrum. For example, let $\alpha_1 = 45°$, $\cos \alpha_1 = \sqrt{2}/2$, for the first observation, the limit wave-length being then at λ_0'; $\alpha_2 = 135°$, $\cos \alpha_2 = -\sqrt{2}/2$ for the second observation, the limit wave-length being then at λ_0''. Then the shift is,[3]

$$\Delta\lambda = \lambda_0' - \lambda_0'' = \frac{\sqrt{2h}}{eV(M_0 - m_0)} \sqrt{\frac{e^2V^2}{c^2} + 2m_0eV}. \qquad (15)$$

We now proceed to calculate the numerical magnitude of this expected shift. We have[4] $h = 6.55 \times 10^{-27}$ erg sec. (p. 41), $e = 1.591 \times 10^{-20}$ abcoulombs (p. 17), $m_0 = 0.899 \times 10^{-27}$ grams (p. 9). $M_H = 1.649 \times 10^{-24}$ grams (p. 9) $c = 3 \times 10^{10}$ cm/sec. It is evident from inspection of formula (15) that the shift is greater the lighter the anticathode and the lower the voltage. We therefore compute curves showing the expected shift under tube voltages below 5 KV, when the following elements are used as anticathodes: lithium (At. Wt. 6.94), beryllium (At. Wt. 9.01) boron (At. Wt. 10.82), carbon (At. Wt. 12.00), sodium (At. Wt. 23.00), magnesium (At. Wt. 24.32), aluminum (At. Wt. 27.10) and silicon (At. Wt. 28.06). The results are shown in Table I.

TABLE I

$\Delta\lambda$ in X-Units (10^{-11} cm.) for the following tube voltages.

Element	0.5 KV	1.0 KV	2.0 KV	3.0 KV	4.0 KV	5.0 KV
Lithium	0.1243	0.0874	0.0610	0.0501	0.0445	0.0378
Beryllium	0.0940	0.0665	0.0470	0.0386	0.0330	0.0291
Boron	0.0784	0.0553	0.0391	0.0321	0.0275	0.0242
Carbon	0.0707	0.0500	0.0353	0.0290	0.0248	0.0219
Sodium	0.0369	0.0261	0.0184	0.0151	0.0129	0.0114
Magnesium	0.0349	0.0247	0.0174	0.0143	0.0122	0.0108
Aluminum	0.0313	0.0222	0.0156	0.0129	0.0110	(<0.01)
Silicon	0.0310	0.0214	0.0151	0.0124	0.0109	(<0.01)

[3]Note added November 4, 1925. After this paper was written it was found that a formula equivalent to the above and based on a similar, though more restricted theory, had already been given by Bronstein, " Zur Theorie des kontinuierlichen Röntgenspektrums " Zeitschrift für Physik, Vol. 32, p. 881, June, 1925.

[4]All numerical values are taken from Sommerfeld's book already quoted in Note 1. Pages are given in parenthesis.

It follows from the above table that the expected shift is very small and, if we take 0.03 X-Units as the limit of detectability, unobservable for elements heavier than aluminum and voltages above 1 KV. Of the elements considered, boron and carbon seem to be the most promising on account of their physical and chemical properties. In the opinion of the writer, the experimental confirmation of the predicted shift would be evidence in favor of the light-quantum theory of radiation at least as strong as that offered by the Compton effect.

It can be readily seen from formulas (13) or (14) that the limit wave-length — or frequency — is practically independent of direction of observation or anticathode material for elements heavier than aluminum and voltages higher than 1 KV. In all these cases, the limit frequency can be calculated with negligible error by the Einstein-Planck formula.

The author owes the idea of the present theory to a lecture on the theory of the Compton Effect delivered by Prof. P. Debye at the Physics Seminar, Massachusetts Institute of Technology, in the early spring of 1925.

October, 1925

ADJUSTMENT OF THE ELASTIC PROPERTIES OF A MODEL KEEL UNITED STATES ARMY AIRSHIP RS-1

By Professor William Hovgaard

I. Introduction

The Airship RS-1 is of the so-called semi-rigid type, and is the largest ship of this kind ever built. It has a volume of 750,000 cubic feet, a length of 282 feet, and a maximum diameter of about 70 feet. It is driven by four motors, capable of developing an aggregate of 1200 h.p., giving a speed of 70 m.p.h. The semi-rigid is intermediate between the non-rigid airship, which consists simply of an elongated gas-bag, and the rigid airship which relies for its strength and stiffness on a complete framework of duralumin girders stiffened by steel wires. It has a keel of duralumin built into the envelope and this girder forms, as it were, the backbone of the ship. (See Fig. 1.) The keel runs the full length of the ship on the underside of the bag. It is a truss girder of equi-

Fig. 1

lateral triangular section with its base continuous with the contour of the envelope and with the apex pointing upwards inside the bag. It is built of thirty-one sections, each one consisting of three longitudinal columns connected by ball and socket joints to transverse triangular frames. The panels so formed are prevented from distorting by diagonal wires. It curves upwards towards the head

and tail and is connected with the top of the envelope by internal suspension wires giving the section of the envelope a cardioidal shape.

The loss of the Italian-built airship *Roma* which occurred while the RS-1 was being designed, caused considerable concern as to the structural strength of the new ship, which was in several essentials of similar type. In 1923 a Special Committee, of which Mr. Henry Goldmark was chairman and of which the author of this paper was a member, was appointed by the National Advisory Committee for Aeronautics to examine and pass upon the design as prepared by the Engineering Division of the Army Air Service. This Committee made an exhaustive study of the design and construction of the RS-1 and made its report in June, 1925.

In considering the problem of longitudinal strength of the ship several important questions arose, which on the design stage could only be solved by means of model experiments. In the spring of 1924 the author of this paper, on the request of the Army Air Service, undertook to design and construct a suitable model and to make the necessary experiments in conjunction with the Air Service. The principal object of these experiments was to investigate the interaction between keel and envelope, in particular when subject to transverse shearing and longitudinal bending in the vertical plane. In a model this could be studied by subjecting it to known shear forces and bending moments, measuring the deflections of keel and envelope separately as well as of the complete model; keel and envelope combined. The model should preferably be so constructed that under similar loading conditions the deflections of ship and model would be in the same ratio as the linear dimensions, but in order to attain this object both in shearing and bending the elastic constants of the model keel had to bear certain definite ratios to those of the ship keel.

Models of non-rigids had been often constructed and the problem of similitude did not in those ships present any difficulty. The gas bag was made of the identical material as in the ship, was placed in an inverted position and filled with water. The internal pressure then increased in a linear ratio from the upper to the lower part of the model, quite corresponding to the increase in gas pressure from the bottom to the top of an actual airship.

The weight of the water took the place of the buoyancy of the gas. Such a model was usually referred to as a " water-model."

As in the tests on non-rigid models, the envelope in the model of RS-1 was made of the same fabric as in the ship, was suspended in an inverted position, filled with water and counterweighted.

Let n be the ratio between the linear dimensions of the ship and the model. Whatever the system of loading, so long as the stresses are within the elastic limit, we want the deflections δ to be proportional to n when the loads are in that proportion. Using the suffix (1) to denote quantities belonging to the model, we have then:

$$\delta_1 = \frac{\delta}{n}. \tag{1}$$

Under these circumstances it is necessary that the tensions in the fabric shall be the same at corresponding points under similar load conditions, since otherwise equation (1) would not be satisfied. Let the principal tensions at any point be T (transverse) and T' (longitudinal), the corresponding principal radii of curvature ρ and ρ', and let p be the difference between the pressures exerted by the fluid in the bag and by the atmosphere outside the bag, hereafter referred to as the net pressure. Then:

$$p = \frac{T}{\rho} + \frac{T'}{\rho'}$$

and since the tensions are the same and $\rho = n\rho_1$, $\rho' = n\rho_1'$ we have:

$$p_1 = \frac{nT}{\rho} + \frac{nT'}{\rho'} = np \tag{2}$$

i.e., the net pressure of the water at any point in the model must be n times the pressure of the gas at the corresponding point in the ship.

The net pressure is the sum of two parts, first, a uniform net pressure p_0, which exists at the bottom of the envelope of the ship and the top of the envelope in the model and, second, a pressure due to density difference between the fluid inside the bag and atmospheric air outside the bag.

That is, at any point:

$$p = p_0 + hb$$

where h in the ship is the height above the lowest point of the bag and b is the buoyancy per unit volume due to the density difference. Hence, in order that p_1 shall be equal to np for all values of p_0, we must have:

$$p_{01} = np_0 \quad \text{and} \quad h_1 b_1 = nhb \qquad \text{where } h = nh_1$$

and hence

$$n = \frac{h_1 b_1}{hb} \text{ or } n = \sqrt{\frac{b_1}{b}} \tag{3}$$

that is, *the linear ratio n must be proportional to the square root of the buoyancies*. When helium is used in the ship and water in the model, we have

$$\frac{b_1}{b} = \frac{62.25}{.055}$$

which gives approximately $n = \frac{100}{3} = 33.33$, *i.e., the linear dimensions of the model shall be .03 of those of the ship*. Thus the scale to which the envelope of the model shall be made is determined and it is necessary that the keel shall be made to the same scale.

We shall now determine the relations which must exist between the elastic constants of the envelope as well as of the keel in order that the deflections in bending, shearing and torsion of each of these members shall be proportional to n when under similar load. If these requirements are fulfilled for keel and envelope separately they must be also fulfilled for the compound girder consisting of keel and envelope combined, whatever the degree of coöperation between these members, provided only that they are connected with each other in the same way in the ship and the model.

Deflection under longitudinal bending, whether in the vertical or horizontal plane and whatever be the system of loading, follows the law (see Equation (8)):

$$\delta \propto \frac{Pl^3}{(EI)}$$

or a sum of such terms. Here P is the load, l stands for any linear

dimension, E is the modulus of elasticity, I is the moment of inertia of a cross section and (EI) will be referred to as the flexural modulus.

For the model we have $P_1=\frac{P}{n}$, $l_1=\frac{l}{n}$ whence

$$\delta_1 \propto \frac{P_1 l_1^3}{(EI)_1} = \frac{Pl^3}{n^4(EI)_1} = \frac{(EI)\delta}{n^4(EI)_1}$$

and if now $\delta = n\delta_1$ we must have:

$$(EI)_1 = \frac{(EI)}{n^3}. \tag{4}$$

For twisting of the keel the formula for the total angle of torsion at any point is of the form:

$$\gamma \propto \frac{Pl^2}{(GI_0)}$$

or a sum of such terms, where I_0 is the polar moment of inertia of a transverse section, G is the coefficient of rigidity and (GI_0) is here called the torsion modulus. For the model we have:

$$\gamma_1 \propto \frac{Pl^2}{n^3(GI_0)_1}$$

and if γ_1 is to be equal to γ we must have:

$$(GI_0)_1 = \frac{(GI_0)}{n^3}. \tag{5}$$

For transverse deflection due to shearing, whether in the vertical or the horizontal plane, we have for a uniform distribution of the shear stresses (see Equation (10)):

$$\delta \propto \frac{Pl}{(AG)}$$

or a sum of such terms, where A is the sectional area and (AG) will be referred to as the shear modulus.

$$\therefore \quad \delta_1 \propto \frac{Pl}{n^2(AG)_1} \text{ which is to be equal to } \frac{\delta}{n} \propto \frac{Pl}{(AG)n}$$

$$\therefore \quad (AG)_1 = \frac{(AG)}{n} \tag{6}$$

It is required, then, that the moduli of flexure and torsion, (EI) and (GI_0), both in bag and keel, must be proportional to n^3. For *the bag* the value of E is the same in the model as in the ship, and since the thickness of material is the same (neglecting differences in the seams), we have both I and I_0 proportional to n^3 as they should be. The shear modulus (AG) in the ship and in the model should be proportional to n. This requirement is satisfied in the bag since G is the same in the ship and the model and the sectional area of the envelope is proportional to the circumference, which varies directly as n.

As regards *the keel* the construction must be such that the elastic constants fulfill the same requirements, but this is a problem which can only be solved experimentally as explained in Chapter III.

The internal suspension cables should be so proportioned that their stretch under similar tensional forces will be in the ratio n. If the cables in the model are of the same material as in the ship, the tensions per unit area should be the same and hence the ratio between the sectional areas must be equal to n.

Changes of pressure will cause changes in the length of the bag, but since the tensions are the same in the ship and the model, the elongations and contractions will be proportional to n. The model keel cannot be adjusted to have proportionate longitudinal strains under similar longitudinal loads, but as these strains are very small relative to those of the bag and as they do not influence the transverse deflections appreciably, this requirement is here neglected.

The flexural modulus of the fabric for bending out of its own plane should for similarity be as n^2 in the ship and the model, but actually it is the same, since the fabric is identical. While this discrepancy may slightly affect the form of the transverse sections, it is not believed to have any sensible influence on the transverse deflections. Also the seams in the model, although made as light as possible, are relatively much heavier than in the ship and will probably increase the same flexural constant somewhat.

The weight of the water in the model corresponds to the lift of the gas in the ship, whence the downward pull of all the weights

in the ship, including keel and envelope must, in the model, be replaced by a similar upward pull. This is produced by a system of counterweighting, which must be very approximately so devised as to give a distribution of forces similar to that of the weights in the ship. Moreover, since the weights of the model keel and the model bag with all their fittings act in a downward direction, and since nothing corresponding to these forces exists in the ship, their action must be completely annulled by counterweights. The whole system of counterweights must be so designed as to offer as little friction as possible to deflections of the model.

The requirements to be fulfilled in the model in order that the experimental results shall apply to the ship, have been here obtained simply and directly. They may also be obtained by Riabouchinsky's Method of Dimensions,[1] as explained by Dr. L. B. Tuckerman of the Bureau of Standards in a paper published by the National Advisory Committee for Aeronautics.

The problem of constructing a keel with the desired elastic properties was entirely new and presented considerable difficulties. To build the keel to scale of duralumin was out of the question; and it was decided to make it of wood, which promised to furnish the simplest and least expensive solution.

In accordance with the foregoing, the model was made practically similar to the ship in the ratio 3 to 100. The most important deviations from similarity were:

a. The fabric of the model bag was of the same thickness as in the ship instead of being of $1/n$th the thickness. Thus the seams came to occupy a larger fraction of the circumference than in the ship, the fabric offered greater resistance to bending in planes normal to its surface and the sectional area was somewhat greater than it should be according to similarity.

b. The keel, being constructed of wood instead of duralumin columns and wires was necessarily of different scantlings, but the outside triangular contour of its sections was everywhere of appropriate dimensions and the fittings for attachment of envelope and suspension wires were in the correct position relative to the enve-

[1]See Dr. E. Buckingham: "Notes on the Method of Dimensions" (The II Theorem), Phil. Mag., Vol. XLII, November, 1921.

lope. The elasticity of the keel was adjusted as explained hereafter in accordance with the formulas given above.

c. The internal suspension cables, connecting the top of the envelope to the top of the keel, were heavier than required by theory, as it was considered desirable to preclude all possibility of accidental breakage when handling the water model. Actually the stretch of the suspension cables is of the second order as compared with the deflections of the model, so that strict similarity is unessential in this case.

The preparation of the model keel, and in particular the development of the formulas for calculating the deflections under various modes of loading of the ship keel as also the experimental adjustment of the elastic constants of the model form the main subject of this paper. Those experiments were ancillary to the more extensive tests of the bag and of the assembled complete model and their analysis, the account of which in detail is not here given as it would be too voluminous and largely outside the scope of this Journal. The last chapter contains, however, a summary of the results and conclusions.

Before the adjustment of the model keel could be made, it was necessary to calculate the deflections of the full-size keel under various conditions of loading.

II. Development of Formulas for Calculating the Deflections of the Keel of RS-1.

The deflections are calculated by a generalized method, regarding the keel as a unit curved girder of varying sectional area and moment of inertia. The keel, being held in the vertical plane, is assumed to be clamped down firmly in a strictly horizontal position between frames 11 and 12, which are at once in the lowest position and of maximum dimensions so that each part of the keel, the after or A-end and the forward or B-end, may be regarded as independent cantilever girders. The loads are in all cases applied at the ends in such a way that the bending moments at 11 and 12 are exactly equal, precluding any tendency to rotation of the section 11–12 in the plane of bending.

The calculations comprise the following cases:

1. Vertical end load. Deflections due to bending.
2. Vertical end load. Deflections due to shearing.
3. Pure couple applied at the ends in the vertical plane. Deflections due to bending.
4. Transverse horizontal end load. Deflections due to bending.
5. Transverse horizontal end load. Deflections due to shearing.
6. Transverse horizontal end load. Deflections due to torsion.

By summation of the deflections under 1 and 2, the total deflections under vertical end load are obtained. Similarly, summation of the deflections under 4, 5, and 6 gives the total deflections under transverse load.

As stated above, the ship keel consists of three longitudinal girders, constructed of duralumin columns, connected to each other and to transverse triangular frames by ball and socket joints. The transverse frames are likewise built of duralumin columns and the panels so formed are stiffened by diagonal wires, which provide resistance to shearing. The frames are about 10 feet apart.

In calculating the moment of inertia of a transverse section of the keel, we assign to each longitudinal column an "effective" sectional area derived from full-scale compression tests:

$$a = \frac{P}{eE} \tag{7}$$

where E is the modulus of elasticity of duralumin and e is the observed compression per unit length.

The diagonal wires are assumed to be just taut when the keel is in the unloaded condition, so that when a panel is subject to shearing one of the two wires becomes slack. Hence, in the calculations we reckon only one diagonal in each panel to be effective.

The origin for the A-end is at frame 12 in the neutral axis of the section and for the B-end at the corresponding point of frame 11. The origins are denoted by O_A and O_B. See Fig. 2. The abcissae are reckoned positive towards the ends of the keel,

and the ordinates positive upwards. The loads are supposed to be applied at frames 1 and 27 at points in the intersection between the center line plane and the base of the triangular frames. We

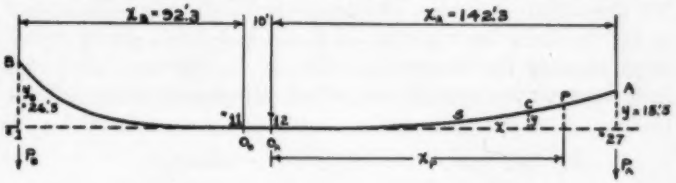


FIG. 2

reckon the neutral axis to extend from its points of intersection B and A with the lines of vertical load on frames 1 and 27. The length along the girth of the neutral axis from the origin to any point C is denoted by s.

The end loads are denoted by P_A and P_B, and the ordinates of their points of application A and B, are x_A, y_A and x_B, y_B respectively. The moment of inertia of a section of the keel normal to the neutral axis is denoted by I. We make $P_A x_A = P_B x_B$.

In the forward or B-end of the keel the frames are nearly

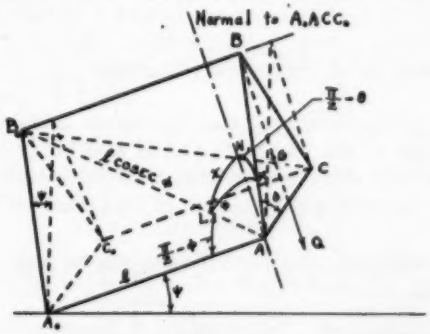


FIG. 3

normal to the neutral axis, but in the entire A-end they are vertical and form an oblique angle with that line. We now substitute

for the keel an ideal structure where all the frames stand normal to the neutral axis and although the panels actually taper a little towards the ends, we shall reckon each of them to be a rectangle. We thus obtain a section between every two frames as illustrated in Fig. 3, where the triangles $A_0\ B_0\ C_0$ and ABC, drawn in full lines, indicate the assumed position of the frames, the dotted lines represent the same frames in their actual positions in vertical planes.

1. *Vertical End Load. Deflections due to Bending.*

At any point P, see Fig. 2, in the A-end the vertical deflection due to bending is:

$$\Delta y_P = -\frac{P_A}{E}\int_0^P \frac{(x_A - x)(x_P - x)}{I} ds$$

Since we want to know the deflection at a number of points along the keel it would be necessary, if this integral were used as it stands, to plot the same number of curves and to integrate each of them. This can, of course, be done by an ordinary planimeter or integrator, but the work can be much shortened by means of the integraph when the above integral is transformed into two in which the value of x_P does not occur under the integral sign:

$$-\Delta y_P = \frac{P_A}{E}\int_0^P \frac{(x_A - x)^2}{I} ds - (x_A - x_P)\frac{P_A}{E}\int_0^P \frac{(x_A - x)}{I} ds \quad (8)$$

By plotting two curves for the expressions under the integral signs in (8) up to the terminal point A and integrating by the integraph, we can obtain the value of these integrals at any point P by simply measuring the ordinate at that point on each of the curves.

A similar expression and corresponding curves may be obtained for the B-end.

2. *Vertical End Load. Deflections due to Shearing.*

The lower face of the keel forms an angle ψ with the horizon and, since the end load P acts in a vertical plane, we have the shear force normal to the neutral axis

$$Q = P\cos\psi$$

In Fig. 3 suppose the frame to the left, $A_0\ B_0\ C_0$, to be held fixed and Q to be applied downwards in the plane of the frame $A\ B\ C$ at its intersection with the neutral axis, G. It is required to find the vertical deflection of this point.

Let l = length of the frame space parallel with the neutral axis.

θ = angle between the side panels and the bottom panels, which angle in the present case is equal to 60 degrees throughout the keel.

ϕ = angle between the diagonals that become effective in the sloping panels just inside the frame under consideration and the transverse sloping frame, in the present case B_0AB.

E_W = modulus of elasticity of the diagonal wires.

a = sectional area of diagonal wires which are effective just inside the frame under consideration.

Δn = deflection of the frame $A\ B\ C$ in its own plane and hence normal to the plane of the lower panel. It may be defined as the normal deflection on a length l of the neutral axis.

In Fig. 3, $A\ N$ is normal to the lower panel and hence in the direction of the deflection Δn; it forms an angle x with the diagonal B_0A. The force Q is resisted by the tension in the two diagonals B_0A and B_0C, while the two other diagonals A_0B and C_0B (not shown in the figure) become slack. Let t be the tension per unit area of wire,

$$\therefore \quad Q = 2ta\cos x$$

As seen from the spherical triangle LNM, in which M is a right angle

$$\cos x = \cos\phi \sin\theta$$

Approximately: $t = \dfrac{\Delta n \cos x}{l \operatorname{cosec}\phi} E_W$ from which

$$\Delta n = \frac{Ql}{2aE_W \sin^2\theta \cos^2\phi \sin\phi} = \frac{Pl\cos\psi}{(AG)}$$

where G is the coefficient of rigidity and A stands for what would be the sectional area in a solid girder. Thus (AG) is the shear modulus, the quantity that ordinarily expresses the stiffness in

shearing, just as (EI), the flexural modulus, expresses the stiffness in bending. In the present case the equivalent for (AG) is:

$$(AG) = 2a E_W \sin^2\theta \cos^2\phi \sin\phi \tag{9}$$

Since $\Delta y = -\Delta n \cos\psi$

we find by integration from the origin to any point P, approximately:

$$\Delta y_P = -P \int_0^P \frac{\cos^2\psi}{(AG)} ds \tag{10}$$

3. *Pure Couple in the Vertical Plane. Deflections due to Bending.*

A couple M is applied in the positive direction at each end of the keel acting in the vertical plane. At any point P in the A-end the vertical deflection is:

$$\Delta y_P = \frac{M}{E}\int_0^P \frac{(x_P - x)}{I} ds = \frac{M}{E}\int_0^P \frac{(x_A - x)}{I} ds - (x_A - x_P)\frac{M}{E}\int_0^P \frac{ds}{I} \tag{11}$$

and a similar expression holds for the B-end.

4. *Transverse End Load. Deflections due to Bending.*

Horizontal transverse forces P_A and P_B are applied at the ends of the keel so adjusted as to produce the same bending moments at frames 12 and 11, $\therefore P_A x_A = P_B x_B$.

It is desired to determine the deflection Δz normal to the vertical

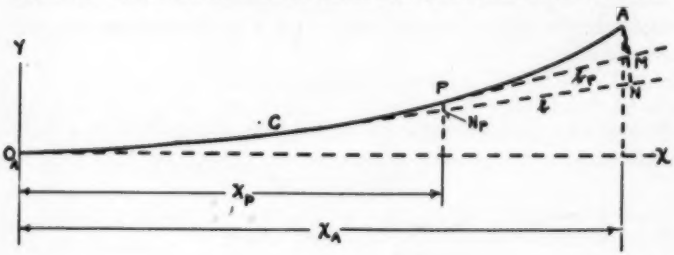


FIG. 4

center-line plane at a certain point P, say in the A-end of the keel. See Fig. 4. Let:

$b = CN$, be the length of the tangent at any point C, inside P, reckoned from C to the foot of the normal on it from A.

$b_P = N_P N$ is the length of the same tangent between the perpendiculars on it from P and A.

In order to make b_P dependent on P only we assume as an approximation:

$$b_P = \frac{PM + (x_A - x_P)}{2}$$

where PM is the length of the tangent at P reckoned from P to the foot of the perpendicular on it from A.

Then the bending moment at C is $M = bP_A$, and the deflection at P is approximately:

$$\Delta z_P = \int_0^P \frac{b(b-b_P)P_A}{EI} ds = \frac{P_A}{E}\int_0^P \frac{b^2}{I} ds - \frac{P_A b_P}{E}\int_0^P \frac{b}{I} ds \quad (12)$$

where I is the moment of inertia about a vertical axis[1] through the center of gravity of the normal section at C. Here b and b_P can be obtained from the plans, and since b_P is approximately a constant in the calculation for any given point P, it is placed outside the integral sign permitting the use of the integraph.

A formula similar to (12) is obtained for the B-end.

5. *Transverse End Load. Deflections due to Shearing.*

Fig. 5 shows again a section of the keel where the frames are assumed to be normal to the neutral axis. We imagine the frame

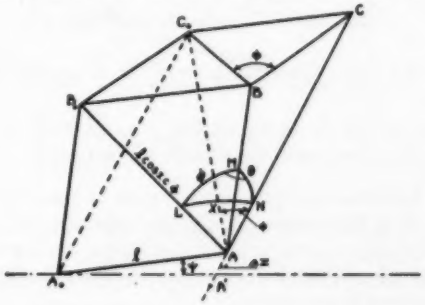


FIG. 5

[1] That is an axis in the vertical plane.

$A_0B_0C_0$ to be held fixed and ABC to be displaced horizontally under the action of a transverse force P_A or P_B. The point A goes to A' and $AA' = \Delta z$. B and C move out the same distance, since shearing is considered alone. The diagonals B_0A, C_0B and C_0A are elongated, and their tension counteracts the displacement.

It is assumed that the angle between the diagonals and the adjacent side of frame ABC is in all three panels equal to ϕ, which is found as the mean value of the angles in the three panels as given on the plans. The angle between a diagonal in one of the sloping panels such as B_0A and the horizontal transverse column CA, along which the displacement takes place, is called x. The angle $BAC = \theta$. In the spherical triangle LMN, Fig. 5, M is a right angle by assumption and we have

$$\cos x = \cos\phi\cos\theta.$$

The diagonal B_0A is elongated by an amount $\Delta z\cos x = \Delta z\cos\phi\cos\theta$. The tension per unit area in B_0A is:

$$t = \frac{\Delta z\cos\phi\cos\theta}{l\operatorname{cosec}\phi} E_W.$$

Since B deflects the same amount as A, the diagonal C_0B comes under the same tension as B_0A and together they offer a resistance to the shear force equal to:

$$2at\cos x = \frac{2\,\Delta z}{l} aE_W\cos^2\theta\cos^2\phi\sin\phi$$

where a is the mean sectional area of the diagonals in the sloping panels.

In the lower panel, in which the area of each of the wires is a_L, we find the transverse component of the tension of the diagonal to be:

$$t_La_L\cos\phi = \frac{\Delta z}{l} a_LE_W\cos^2\phi\sin\phi.$$

Equating the sum of the components of the tensions in the diagonals to the transverse force, we find:

$$P = \frac{\Delta z}{l} E_W\cos^2\phi\sin\phi(2a\cos^2\theta + a_L)$$

from which the deflection for one section is found:

$$\Delta z = \frac{Pl}{E_W \cos^2\phi \sin\phi (2a\cos^2\theta + a_L)}$$

or approximately, for any point P relative to the origin:

$$\Delta z_P = \frac{P}{E_W} \int_0^P \frac{ds}{(2a\cos^2\theta + a_L)\cos^2\phi \sin\phi} \tag{13}$$

6. *Transverse End Load. Deflections due to Torsion.*

While the diagonal wires are neglected in bending, the longitudinal columns are neglected in torsion. In order to simplify, we take as the area of any one wire the mean area of all the six wires in the three panels of a frame space, and ϕ is again the mean of this angle in all three panels as given on the plans.

In Fig. 6 let C be any point in the forward part of the neutral axis of the keel, distant s from O_B. Draw the tangent to the neutral axis at C and drop $BN = a_B$ normal to it, drop also $PN_P = a_P$ normal to it from a point P, the deflection of which, Δz, we want to determine.

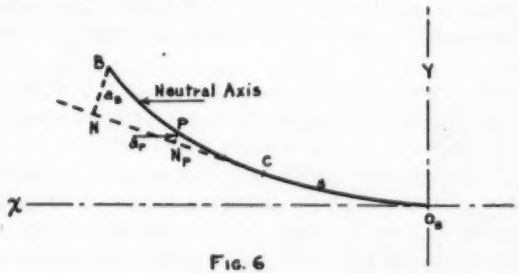


FIG. 6

In Fig. 7 we assume frame $A_0B_0C_0$ to be fixed and a twisting moment to be applied to ABC causing it to turn in its own plane through an angle of torsion $l\gamma$. Then γ is the angle of torsion per unit length and the total torsional angle of a point P in Fig. 6 relative to the origin is approximately

$$\gamma_P = \int_0^P \gamma ds$$

while the transverse deflection at that point is

$$\Delta z_P = \int_0^P a_P \gamma ds. \tag{14}$$

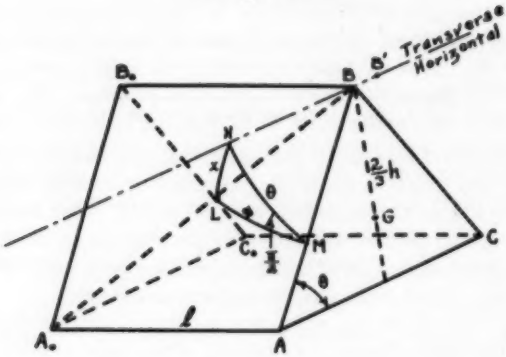


FIG. 7

We neglect deflections in the vertical plane, which within the elastic limit is permissible to the first order. The value of a_B can be measured for any point from the plans and γ is determined as follows:

Due to the twisting motion, each of the points A, B, C, in Fig. 7, moves in a direction normal to the corresponding vector from the center of gravity G of the triangle. For instance, the point B goes to B', swinging out horizontally in an arc with radius $GB = \frac{2}{3}h$, where h is the height of the triangle. Hence the diagonal A_0B is elongated by an amount $BB'\cos x = \frac{2}{3}h\gamma l\cos x$, where x is again the angle between the diagonal and the direction of the displacement, here BB'.

The angle x is found from the spherical triangle LMN, Fig. 7, where M is a right angle.

$$\therefore \quad \cos x = \cos\theta\cos\phi.$$

The elongation of A_0B produces a tension per unit area in the wire:

$$t = \frac{2}{3}\frac{h\gamma\cos x}{\operatorname{cosec}\phi}E_W = \frac{2}{3}h\gamma\cos\theta\cos\phi\sin\phi E_W \tag{15}$$

and the same tension is produced in the corresponding diagonal in the two other panels. The components of the tensions of the three diagonals normal to the vector from G have a total moment about that point, which must be equal to the twisting moment at the frame station.

Referring to Fig. 6, the twisting motion is produced by the horizontal transverse force P_B at B, acting with a leverage which at any point C is equal to a_B. It follows that:

$$P_B a_B = 3ta\cos x \times \tfrac{2}{3}h = 2tah\cos\theta\cos\phi.$$

Substituting the value of t from (15) and solving for γ we find:

$$\gamma = \frac{3P_B a_B}{4ah^2 E_W \cos^2\theta\cos^2\phi\sin\phi}. \tag{16}$$

Substituting in (14) we find:

$$\Delta z_P = \frac{3}{4}\frac{P_B}{E_W}\int_0^P \frac{a_P a_B}{ah^2\cos^2\theta\cos^2\phi\sin\phi}\,ds. \tag{17}$$

For the end point B this becomes:

$$\Delta z_B = \frac{3}{4}\frac{P_B}{E_W}\int_0^B \frac{a_B^2}{ah^2\cos^2\theta\cos^2\phi\sin\phi}\,ds. \tag{17'}$$

A similar formula is obtained for the A-end. In evaluating the integral of (17), it is necessary to construct separate curves for each value of P, because a_P occurs under the integral sign.

By means of the formulas developed in this and in the previous sections the deflections were calculated for certain couples and for loads applied vertically and horizontally at the ends.

The numerical calculations which necessitated the construction of curves and their integration by means of the integraph were very extensive and shall not here be given, but the results, reduced to the scale of the model by multiplication with 3: 100, are given in Table II below, where they are compared with those obtained experimentally with the model keel as finally adjusted.

III. Construction and Adjustment of the Model Keel.

The problem of constructing the model keel was especially difficult on account of the relatively small rigidity of the actual keel in shearing, the diagonal wires from which the shearing strength was entirely derived being very light relative to the columns of the longitudinal and transverse members. It was required to produce a curved girder of certain contour dimensions with a certain modulus of flexure (EI) and a certain modulus of shear (AG), where the ratio between (EI) and (AG) was much greater than found in any known material. It was necessary, therefore, while retaining the external triangular contour of the sections at every point so to lighten up or rout out the section of the keel as to reduce the modulus of shear much more than the modulus of flexure.

Mahogany was chosen as material for the model keel on account of its freedom from warping and its great homogeneity. In view of the strong curvature of the keel it was necessary to construct the model of thin boards bent over a mold to the correct shape and glued together, but before deciding on the best way of lightening it, it was necessary to make a number of preliminary tests on straight and curved pieces, at first solid and gradually lightened so as to study the effect on the modulus of flexure and shear.

A. Preliminary Tests

It was desired first to determine the true or fundamental elastic constants, the ordinary modulus of elasticity E and the coefficient of rigidity G, since a knowledge of these would facilitate the adjustment of the more complex quantities (EI) and (AG) for various shapes of section. A knowledge of G was, in particular, desirable because the shearing deflections were so great as to form an appreciable fraction of the total deflections.

The tests were made in the Testing Materials Laboratory of the Massachusetts Institute of Technology with the assistance of Mr. R. G. Adams of the Department of Mechanical Engineering.

Guided by some preliminary tests on small pieces, a straight beam, 60 inches long, of uniform triangular equilateral section with sides $s = 4$ inches was constructed of a number of well seasoned mahogany boards glued together.

As first made, the beam was of solid section and was known to possess much greater stiffness than required. Notably the ratio (EI) to (AG) was much greater than required in the model of the ship keel, but the tests with full section furnished the fundamental values of E and G.

The beam was supported at two points and was loaded at the middle. The tests were made in a number of series under different loads, each series comprising three spans, 54 inches, 36 inches and 18 inches. The beam was set up in an Olsen Testing Machine, the supports resting on a 6-inch I-beam of steel. The load was applied through a wood block which fitted snugly over the apex and down the sides of the beam so as to prevent splitting or local crushing. For each span a number of different loads were applied, varying from 50 pounds up to from 200 to 500 pounds. The deflection at the middle was observed on an Ames' dial. As it was found that the deflections, being well within the elastic limit of the material, were very nearly proportional to the loads for any given span, the mean value corresponding to a deflection of 100 pounds was calculated for each span.

The deflection in bending under a concentrated load in the middle for a beam firmly supported at the ends is given by

$$\delta_B = \frac{Pl^3}{48EI} \tag{18}$$

where l is the length between supports, P is the load.

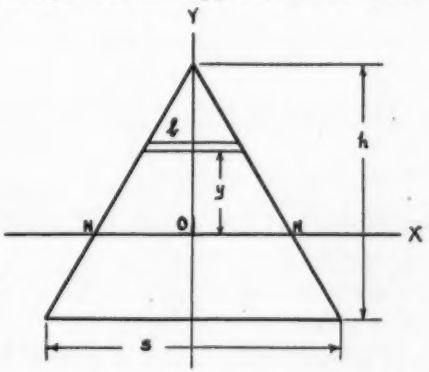


FIG. 8

The deflection in shearing was determined by equating the internal elastic work to the external work done by the load. Assuming the shear stresses to be distributed parabolically as usual we have the shear stress at any point (Fig. 8).

$$q=\frac{p}{2}\frac{m}{Ib} \tag{19}$$

where m is the moment of the sectional area above the point about an axis through that point parallel with the neutral axis, b is the breadth of an elemental strip at the point. The elastic work is

$$l\int_{-\frac{h}{3}}^{+\frac{2}{3}h}\int_{-\frac{b}{2}}^{+\frac{b}{2}}\frac{q^2}{2G}dxdy \tag{20}$$

where h is the height of the triangle.

This must be equal to $\frac{1}{2}P\delta$, where δ is the deflection at the middle. For a solid triangular section, we have:

$$m=\int_y^{\frac{2}{3}h} ybdy=\frac{s}{3h}\left[\frac{4}{27}h^3-hy^2+y^3\right]. \tag{21}$$

Substituting in (19):

$$q=\frac{P}{6I}\left(\frac{2}{9}h^2+\frac{1}{3}hy-y^2\right) \tag{22}$$

whence from (20) we find:

$$\frac{sP^2lh^5}{18\times 240\,GI^2}=\frac{1}{2}P\delta_s$$

but $I=\frac{1}{18}Ah^2$, where $A=\frac{1}{2}hs$ is the area of the triangle.

$$\therefore\quad \delta_s=\frac{6}{5}\frac{Pl}{4AG}. \tag{23}$$

If the shear stresses had been uniformly distributed we should have had

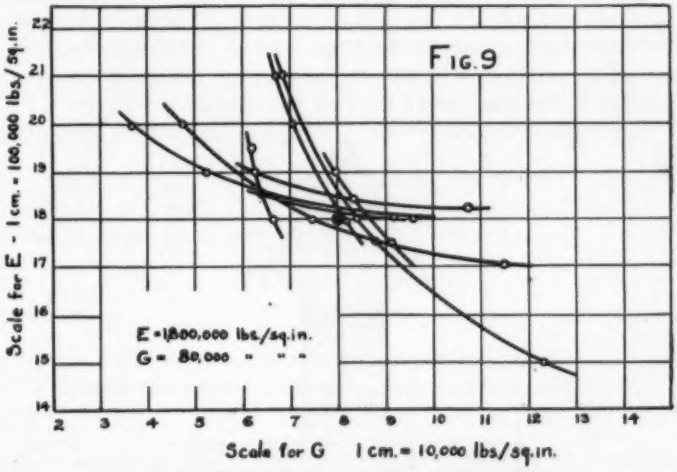
$$\delta_s=\frac{Pl}{4AG}.$$

Hence the form factor with this mode of support is $\frac{6}{5}$.

The complete formula for the deflection is:

$$\delta = \frac{Pl^3}{48\,EI} + \frac{6}{5}\,\frac{Pl}{4\,AG}. \tag{24}$$

From the tests the value of δ was found for three different spans corresponding to an average load of 100 pounds and thus three equations such as (24) were obtained, from any two of which E and G could be found, while the third equation furnished a check or served to give a better average value. The following equations were obtained as an average of the last series of experiments of this kind, during which the load ranged from 50 to 500 pounds.



The various quantities are entered in the same order of sequence as in Equation (24).

$$.0428 = \frac{1}{48} \times \frac{100 \times \overline{54}^3}{E \times 4.62} + \frac{6}{5} \times \frac{100 \times 54}{4 \times 6.93G}$$

$$.0143 = \frac{1}{48} \times \frac{100 \times \overline{36}^3}{E \times 4.62} + \frac{6}{5} \times \frac{100 \times 36}{4 \times 6.93G}$$

$$.00243 = \frac{1}{48} \times \frac{100 \times \overline{18}^3}{E \times 4.62} + \frac{6}{5} \times \frac{100 \times 18}{4 \times 6.93G}.$$

From these and previous experiments curves were constructed as shown on Fig. 9 and the following values were finally adopted:

$$E = 1{,}800{,}000 \text{ pounds/square inch.}$$
$$G = 80{,}000 \text{ pounds/square inch.}$$

It was now possible to form an opinion of the nature and extent of the lightening of the beam section required according to the law of similitude, and the section of the beam was modified by cutting out material on the sides as indicated on Fig. 10, leaving a web, which to begin with was of 1¼-inch thickness and 1-inch height, but which was gradually reduced in thickness to 1 inch and increased in height to 1½ inches. As, in spite of these changes, the shear modulus was still too large, holes of 1⅛ inches diameter, spaced 2 inches from center to center, were bored in the web and finally the thickness of the web was reduced to ⅝ inch.

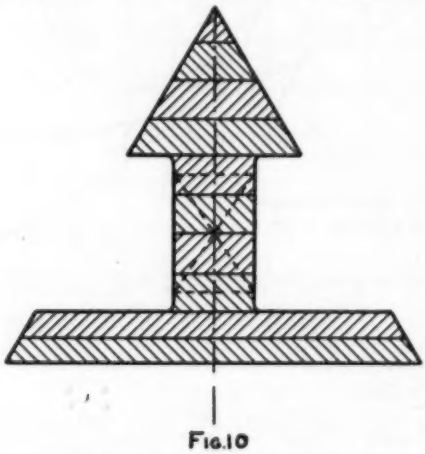


FIG. 10

Under these circumstances unknown form factors, K_1 and K_2 had to be applied to I and A in formula (24), which took the form:

$$\delta = \frac{Pl^3}{48(K_1EI)} + \frac{Pl}{4(K_2AG)}. \qquad (25)$$

The values of I and A are here those corresponding to the actual sections of the beam disregarding the holes in the web, and the influence of the holes as well as of the special form of the section on the distribution of the stresses is taken care of by introducing the factors K_1 and K_2. The values of E and G are the fundamental ones found from the full-section beam. There is, however, no necessity for determining K_1 and K_2 separately; it is sufficient to deal with the quantities (K_1EI) and (K_2AG) as entities, referred to as the moduli of flexure and shear. These moduli are independent of the form of loading, but depend on the form and area of the section and hence on the distribution of the stresses and on the elastic properties of the material.

Roughly, in order to comply with the requirements of similitude, it was desired to make (K_1EI) equal to $\frac{1}{n^3}$ of the same quantity for the ship, and (K_2AG) equal to $\frac{1}{n}$ of that quantity for the ship.

The following table gives as a sample the results of the tests when the web was 1 inch thick and $1\frac{1}{2}$ inches high, while the web was lightened by 27 holes of $1\frac{1}{8}$ inches diameter.

TABLE I

Tests on a straight, triangular beam, 4×4×4 inches
Section similar to Fig. 10 and lightened by holes in the web.

Span	54″		36″		18″	
	P *lbs.*	*in.*	*P* *lbs.*	*in.*	*P* *lbs.*	*in.*
	0	0	0	0	0	0
	30	.0218	50	.0137	50	.0040
	50	.0338	100	.0255	100	.0070
	100	.0650	200	.0500	200	.0136
	200	.1257	0	0	300	.0200
	0	0			400	.0259
					0	0
Sum.....	380	.2463	350	.0892	1050	.0705
Average..	100	.0648	100	.0255	100	.00671

From these data the following equation is obtained for 54-inch span and similar equations for 36-inch and 18-inch span:

$$.0648 = \frac{100 \times 157{,}500}{48(K_1EI)} + \frac{100 \times 54}{4(K_2AG)}.$$

These furnish the mean values of the moduli:

$$(K_1EI) = 6.69 \times 10^6, \quad (K_2AG) = 8.58 \times 10^4.$$

The corresponding values calculated for the maximum section of the ship's keel were: $232{,}000 \times 10^6$ and 132,000, which if multiplied respectively by $\frac{1}{n^3}$ and $\frac{1}{n}$ or by 27×10^{-6} and .03, give the values:

$$(K_1EI) = 6.265 \times 10^6$$
$$(K_2AG) = 5.28 \times 10^4$$

showing that the moduli arrived at in the straight beam were both of them too large, and the shear modulus relatively more so than the bending modulus.

The results gave, however, sufficient guidance for the preparation of the model keel.

The preliminary tests extended over about one month and comprised eighteen series.

B. Tests with the Model Keel

The keel was constructed in the model shop of the Massachusetts Institute of Technology and was built up of ten boards of mahogany. As first tested the sides at frames 11 and 12 were $4 \times 4 \times 4$ inches, tapering towards the ends in the same proportion as in the ship keel. The sides were cut out similarly to those of the 60-inch beam used in the preliminary tests and illustrated in Fig. 10. The web was at first ¾ inch throughout the length and had a height of 1½ inches amidship, tapering toward the ends in the same proportion as the sides. The thickness of the lower flange was ½ inch. As regards the triangular contour of the sections, geometrical similarity with the ship keel was preserved throughout. The exact design of the boards, from which the model was made, was prepared by Prof. H. H. W. Keith of the Massachusetts Institute of Technology. The construction of the keel

with its varying section and strong curvature was a task of great difficulty due to the extreme accuracy required, but was successfully accomplished thanks to the great skill and industry of the model maker, Mr. George O'Hara. When the keel was completed it deviated less than .2 inch from its correct form at the ends and seemed to be entirely free from warping. As first tested, there were no holes in the web of the model.

The keel was clamped down between frames 11 and 12 under a pressure of from 1000 to 2000 pounds in a testing machine in

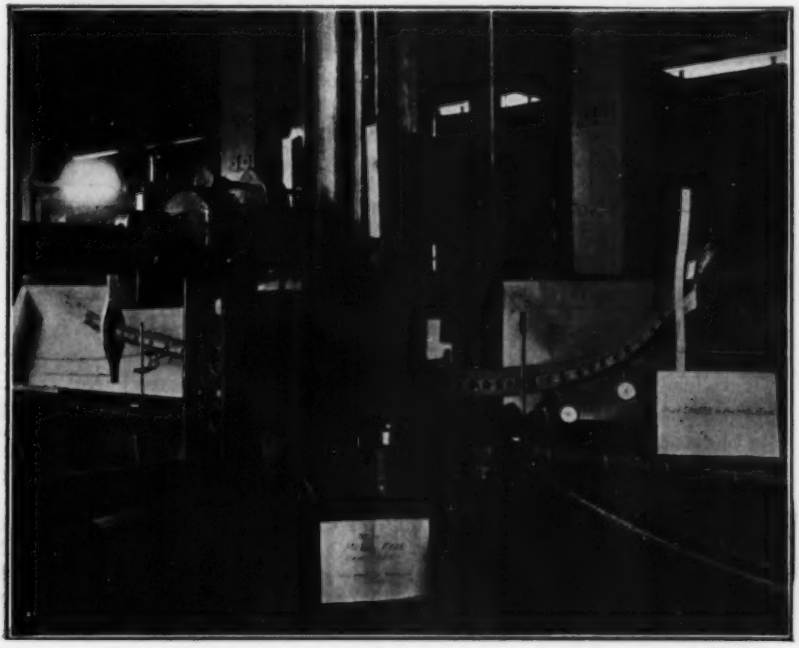


Fig. 11

the Testing Materials Laboratory. (See Fig. 11.) The loads were applied at the ends in the same location and in the same manner as assumed in the calculations for the ship keel, but were reduced

in the ratio 3 to 100. It follows that the bending moments at frames 11 and 12 were again equal and of opposite sign. The tests took place at the end of June and the first two weeks of July, 1924. In all twenty-one tests were carried out, after each of which adjustments were made by cutting holes in the web and by reducing the width of the top and bottom flanges. As finally adjusted, the sides of the keel at frames 11 and 12 were 3.6 inches in length and there were 41 holes in the web, varying in diameter from $\frac{5}{8}$ to $1\frac{1}{8}$ inches.

The measurement of the deflections was made by Ames' Dials pressing against the horizontal face of small wedge-shaped chocks of wood which were glued on under the bottom of the keel at each of the points where deflections were taken. The dials permitted readings to an accuracy of better than .001 of an inch, and were probably reliable to 1 or 2 thousandths of an inch.

The vertical end loads were:

$P_A = 15.00$ pounds corresponding to 500 pounds in the ship.
$P_B = 23.13$ pounds corresponding to 771 pounds in the ship.

The loads were so applied that the line of force passed through the middle point of the lower side of the keel on frames 1 and 27. The bending moment at frames 11 and 12 was 768 inch pounds, corresponding to about 70,000 foot pounds in the ship.

The pure couples were applied by fitting a vertical cross bar of total leverage 12.8 inches at each of frames 1 and 27. (Fig. 11.) Forces of 20.5 pounds were applied at the upper and lower end of these levers, producing couples of 262.4 inch pounds corresponding to about 24,000 foot pounds in the ship.

The transverse loads were applied to lines leading off to the same side from frames 1 and 27 and taken over pulleys. The line at the A-end carried 3.25 pounds and at the B-end 5.00 pounds, giving bending moments of 166.2 inch pounds at frames 11 and 12, corresponding to about 15,500 foot pounds in the ship.

Table II gives a comparison between the final observed values of the deflections of the model keel and the calculated values.

TABLE II

Summary of Calculated and Observed Deflections of Model Keel
Δy *and* Δz

End	Frame No.	Vertical Load		Pure Couple		Transverse Load	
		Calc.	Obs.	Calc.	Obs.	Calc.	Obs.
		in.	*in.*	*in.*	*in.*	*in.*	*in.*
B	1	.093	.093	.040	.045	.283	.201
	3	.070	.069	.024	.026	.115	.095
	5	.046	.046	.012	.013	.043	.038
	8	.014	.014	.002	.002	.011	.009
A	14	.006	.006	.002	.002	.004	.004
	18	.029	.030	.008	.010	.015	.015
	21	.068	.067	.019	.027	.032	.032
	24	.016	.114	.041	.050	.053	.054
	27	.162	.170	.082	.097	.082	.085
	$M_{11} = M_{12}$	768 in. lbs.		262 in. lbs.		166 in. lbs.	

It is seen that for end load in the vertical plane the adjustment of the B-end was practically perfect and for the A-end the maximum deviation which was found at frame 27, is .008 inch, showing about 5 per cent greater flexibility.

Under a pure couple the adjustment of the B-end was very good, but for the A-end the flexibility was somewhat too great.

A comparison of the deflections under vertical load, where both shearing and bending are active, with the deflections under a pure couple, where only bending is active, rendered it possible to differentiate between the deflections due to pure shearing and pure bending.

Under transverse load there was a surprisingly close correspondence between the calculated and observed deflections for the A-end, but for the B-end the model showed too great rigidity, amounting for frame 1 to about 30 per cent. The cause of this is to be found mainly in the deflections due to torsion, which at the extreme end of the bow were about 80 per cent of the total. In other words, the B-end of the keel had too great torsional stiffness.

After the tests the model keel showed no sign of weakness or permanent strain. It was believed that the model keel could stand much greater stresses without injury, but it was not found

necessary to apply greater loads, as the keel was strong enough for the contemplated elastic tests.

The model keel, as completed, represented the keel of the ship sufficiently close for the main object of the tests, which was a study of the coöperation of the keel and the envelope. For this purpose complete similitude between the keel of the model and that of the ship was unnecessary and could not be attained with any degree of certainty even with a more elaborate and costly model, because the elastic properties of the ship keel were by no means accurately known.

IV. Summary of Test Results with Keel Alone, Envelope Alone and Complete Model.

The tests with the envelope without and with keel were carried out at Wilbur Wright Field in November, 1924, by the author in conjunction with Lieut. R. E. Robillard, of the United States Air Service, and the results so obtained were checked during the first months of 1925 by Lieutenant Robillard. During the latter tests Lieutenant Robillard devised a new and ingenious method of counter-balancing by which the sand bags hitherto used as counter-weights were replaced by suspension from a scale-beam through a system of levers so adjusted that the correct system of upward forces was obtained. This arrangement is shown in Fig. 12.

The most important tests were those under vertical end load, but tests were made also under horizontal end load. It was desired, moreover, to study the so-called " breathing " deflections and stresses called forth in a semi-rigid by variations in the gas pressures, such as may be caused, for instance, by rapid changes in altitude. When the pressure is altered, the bag suffers deformations which, on account of the taper towards the ends and the consequent differential movement at adjacent sections, produce shearing strains in the fabric. These strains which are particularly great in ships of cardioidal section, produce stresses in the keel. This action was first pointed out, and its magnitude predicted theoretically, by Mr. W. W. Pagon, a member of the Special Committee on the Design of the RS-1. It does not exist in rigid

airships and in non-rigids, where no keel is found; it simply causes deformations in the bag without material stresses.

It was found that under a certain *vertical end load*, referred to as the standard, corresponding to a bending moment at frames 11 and 12 of about 70,000 foot pounds in the ship, the deflections of

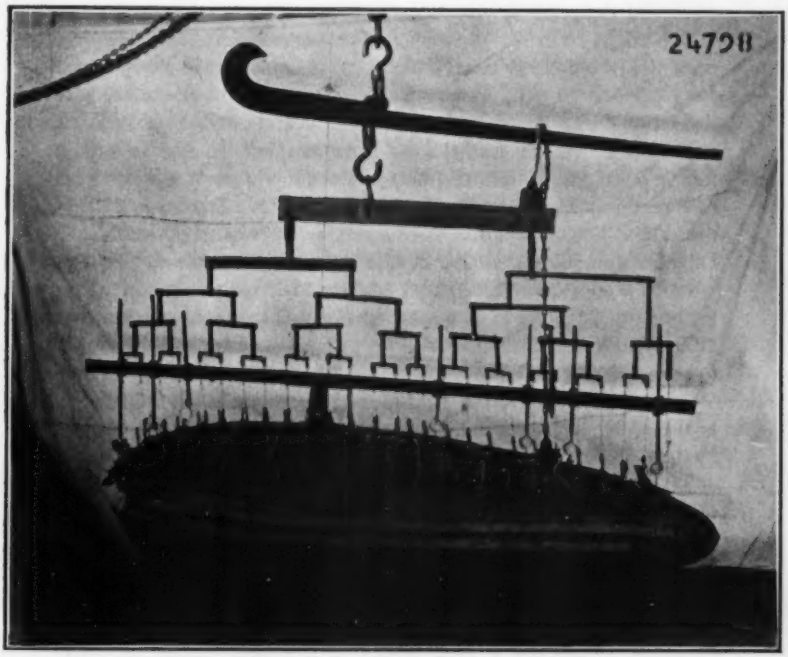


Fig. 12

the complete model, keel and envelope combined, were about one half of those of the keel alone under the same load. This was when the pressure of the bottom at the envelope was about 17 inches of water in the model, corresponding to about ½ inch of water in the ship. It shows that *the keel and envelope shared the burden of the load fairly equally*. It is a necessary condition for such coöpera-

tion that the envelope shall be laced tightly to the keel so as to insure an effective transmission of shear forces.

Comparing the observed deflections of the complete model under vertical end load with those obtained by calculation, it was found that *at 24-inch water pressure the model showed almost perfect coöperation between the envelope and keel.* At low pressures the observed deflections were about 50 per cent in excess of those which might be expected under perfect coöperation. The principle on which the deflection was calculated is explained in this Journal for December, 1923.[3] It enables us to calculate in the usual way deflections and stresses in structures compounded of various materials by simply multiplying their moments of inertia with a factor μ, which is the ratio between the modulus of elasticity E_0 of a certain material adopted as a standard and E that of another material. The application of this method to semi-rigid airships is explained on pages 205–207 in said issue of this Journal.

Even a considerable *deflation of the model*, such as produced by letting say 25 pounds of water, or about 2 per cent of the total volume, out of the bag, *had little influence on the deflections due to vertical load.* It follows that a rapid descent of the ship of some 600 feet under conditions where for some reason the ballonets fail to function, should not materially affect the stresses due to vertical load.

The tests of the envelope alone gave very clear indication of *the value of the envelope as an active member of the structure.* The deflections were only from three to three and one-half times as great as those of the complete model and were little affected by deflation.

The "*breathing*" deflections caused by changes in gas pressures between 0 inch and ½ inch, were of the same order as those produced by a bending moment of about 70,000 foot pounds at frames 11 and 12. At higher pressures and under conditions of partial deflation, where the pressure was zero, the breathing stresses seemed to be insignificant.

The deflections of the complete model *under horizontal end load* were surprisingly small, especially in the bow, as compared with

[3] W. Hovgaard, "A new proof of the theory of ordinary bending and its extension to beams of non-homogeneous materials," Journ. M and P, Vol. II, No. 4, December 1923.

those of the keel alone under the same load, showing again the great aid which the keel receives from the envelope. Under conditions of partial deflation there seemed to be, however, a marked reduction in transverse stiffness of the envelope.

The *stresses* due to bending may be found by the ordinary formula, provided only the moment of inertia is calculated as explained above and then reduced by some 50 per cent to allow for imperfect coöperation between keel and envelope at low gas pressures. The *tensions in the diagonal wires* may be calculated under the assumption that the shear forces acting on the keel are three quarters of what they would be if the keel had to carry the whole load.

The envelope, when lashed tightly to the keel, seems to offer very great resistance to *torsion*, probably more than ten times that offered by the keel at the nose and several times more than that of the keel at the tail.

In order to secure *sufficient strength of the nose cone* it is desirable to adopt a gas pressure at the bottom of the envelope in the ship of not less than one half inch as a standard.

The following table gives the average relative magnitude of the deflections under various conditions of gas pressure for the envelope alone and for the complete model, using in each case the deflection of the bare keel as unity. The table shows also the relation between the calculated and observed deflections.

TABLE III

Deflections of Model

	Calculated	*Observed*		
		24" Pressure	*4½" Pressure*	*25 lbs. of water run out Zero Pressure*
Keel alone.....	1	1	1	1
Envelope alone	2	1½	2	2¼
Complete model	⅜	7/16	⅜	⅜

THE ELEMENTARY THEORY OF ALMOST PERIODIC FUNCTIONS OF TWO VARIABLES[1]

By Philip Franklin

The theory of almost periodic functions of one variable was founded by Harold Bohr.[2] In this paper we generalize the definition so as to apply to functions of two variables, and prove certain properties of these functions. We confine ourselves to the elementary theory, that is properties not depending on the fundamental theorem (the analogue of Bohr's fundamental theorem, or of Parseval's theorem for the trigonometric expansion of doubly periodic functions), reserving for a later paper the proof of the fundamental theorem for our functions and a discussion of some of its consequences. Many of Bohr's theorems for the functions of one variable extend directly, others require more or less modification to apply to our case. For the sake of giving a complete and independent exposition, we shall not refer to the theorems of Bohr in proving our results, but shall derive them all from first principles.

The functions which we discuss will be functions of two *real* variables, which functions may assume *complex* values. That is, our functions are of the form

$$f(x, y) = R(x, y) + iI(x, y),$$

where $R(x, y)$ and $I(x, y)$ are real functions of real variables. We do this merely to simplify certain of our formulas, and enable us to use exponentials instead of trigonometric functions. A theory in which all the variables which enter are real may be easily arrived at from ours by merely taking the real parts of all of our equations. We first define a *displacement vector* (s, t) of a function $f(x, y)$, belonging to ϵ as a pair of numbers such that

$$f(x+s, y+t) - f(x, y) \leq \epsilon$$

for all values of x and y. A function of two variables, x and y,

[1] Presented to the American Mathematical Society October 31, 1925.

[2] Zur Theorie der fast periodischen funktionen, Acta Mathematica, 45: pp. 29–127. (1924.)

continuous for all values of these variables, will be said to be almost periodic if, for every ϵ there is a number r such that, given a circle of radius r anywhere in the x, y plane, there is at least one displacement vector (s, t) having its initial point at the origin, and its terminus s, t somewhere inside this circle.

We note that the choice of a circle in the above definition is somewhat arbitrary, as any figure of finite dimension and fixed shape with size given by a parameter replacing r might have been used. The resulting definitions would all have been equivalent, since any figure of given shape and finite dimension may be included in some circle, and, conversely, any such figure could be taken so large as to include in its interior some circle of given radius r. We could either leave the orientation of the figure arbitrary, or restrict it to be fixed. In many of our proofs it will be convenient to use as the figure a square with sides parallel to the axes, and the definition in the form: for every ϵ, there is an a, such that in any square with sides parallel to the axes of side a, there is at least one displacement vector having ts terminus in this square. We shall call such a number a, a *displacement number* belonging to ϵ, and the radius r of the earlier definition a *displacement radius* belonging to ϵ.

We shall now prove two theorems which bring out some restrictions on the character of almost periodic functions not explicitly stated in the definition.

Theorem I. *Every almost periodic function of two variables is bounded.*

For, let r be a displacement radius for $\epsilon=1$, and M the maximum value of the absolute value of the given function, $|f(x, y)|$, in the circle of radius r with the origin as center, which exists since $f(x, y)$ is continuous. The function $f(x, y)$, in absolute value never exceeds $M+1$. For, since there is a displacement vector belonging to 1 with initial point at the origin and terminus s, t somewhere inside a circle of radius r and center at any point x', y', we have:

$$|f(x+s, y+t)-f(x, y)|\leq 1$$

for all values of x, y. In particular,

$$|f(x', y')-f(x'-s, y'-t)|\leq 1.$$

As $x'-s$, $y'-t$ is inside the circle about the origin with radius r,

$$|f(x'-s, y'-t)| \leq M,$$

and hence

$$|f(x', y')| \leq M+1$$

which proves the theorem.

Theorem II. *Every almost periodic function of two variables is uniformly continuous in the entire plane.*

Let r be a translation radius belonging to $\epsilon/3$ for the given function $f(x, y)$. Let δ be chosen less than r and such that, for any two points in the circle about the origin of radius $2r$, whose distance from one another is less than δ, the function $f(x, y)$ differs by less than $\epsilon/3$. The existence of δ follows from the fact that $f(x, y)$ is continuous, and therefore uniformly continuous, in any closed region. Now consider any two points x', y' and x'', y'' whose distance is less than δ. By the procedure used in Theorem I, we may find a displacement vector s, t which takes the first point into a point a', b' inside the circle about the origin with radius r, and consequently takes the second point x'', y'' into a point a'', b'' inside the circle about the origin with radius $2r$, the distance between these points being less than δ. Hence we have:

$$\begin{aligned}
|f(a', b')-f(a'', b'')| &\leq \epsilon/3,\\
|f(x', y')-f(a', b')| &\leq \epsilon/3,\\
|f(x'', y'')-f(a'', b'')| &\leq \epsilon/3,
\end{aligned}$$

from which it follows that

$$|f(x', y')-f(x'', y'')| \leq \epsilon.$$

As our only hypothesis on the points x', y' and x'', y'' was that they were at distance less than δ, this proves the uniform continuity of $f(x, y)$ in the entire plane.

Corollary. *If s_1, t_1 is a displacement vector of $f(x, y)$ belonging to ϵ_1, there exists a positive number δ depending on ϵ_1 and $\epsilon_2 > \epsilon_1$ such that if $|s_2-s_1| < \delta$, $|t_2-t_1| < \delta$, s_2, t_2 is a displacement vector for ϵ_2.*

For, let δ be such that for any two points at distance less than $\sqrt{2}\,\delta$ the function differs by less than $\epsilon_2-\epsilon_1$. Then:

$$f(x+s_2, y+t_2)-f(x, y)\,| \leq |f(x+s_2, y+t_2)-f(x+s_1, y+t_1)\,| + |f(x+s_1, y+t_1)-f(x, y)\,| \leq (\epsilon_2-\epsilon_1)+\epsilon_1=\epsilon_2.$$

The next few theorems have as their aim the proof that there are certain ways of combining almost periodic functions so as to yield new almost periodic functions.

Theorem III. *Any two almost periodic functions are simultaneously almost periodic, that is, we may find an r such that in each circle of this radius there is a single displacement vector s, t, which belongs to ϵ for both functions.*

Let δ be selected, according to the corollary of Theorem II, so that if s_1, t_1 is a displacement vector for the first function $f_1(x, y)$ belonging to $\epsilon/2$, and $|s_2-s_1\,|<\delta$, $|t_2-t_1\,|<\delta$, s_2, t_2 will be a displacement vector for ϵ. Let r_1 and r_2 be displacement radii belonging to $\epsilon/2$ for the given functions $f_1(x, y)$ and $f_2(x, y)$ respectively. Select a number $a>2r_1$, $2r_2$ and an integral multiple of δ. This will be a displacement side for both functions, that is, in every square of side a with sides parallel to the axes, there will be displacement vectors belonging to $\epsilon/2$ for each of the two functions. Now take N an integer $> 2a/\delta$. We shall show that in every square of side Na with sides parallel to the axes, and hence in every circle of radius $Na/\sqrt{2}$, there is at least one displacement vector belonging to ϵ for both functions.

For, take any such square, and divide it into N^2 squares each a on a side by lines drawn parallel to the axes. In each of these squares we can find two vectors s_1, t_1 and s_2, t_2 which are displacement vectors for $\epsilon/2$ for $f_1(x, y)$ and $f_2(x, y)$ respectively. Now consider the set of vectors s_2-s_1, t_2-t_1 all drawn so that they have one extremity at the origin. Their other extremities all lie in a square of side $2a$ with center at the origin. Now divide this into squares δ on a side and hence $4a^2/\delta^2$ in number by lines drawn parallel to the axes. Since there are N^2 vectors, and

$$N^2>4a^2/\delta^2$$

there must be at least one pair of vectors having their extremities in the same square. Let this pair be S_2-S_1, T_2-T_1. S_2, T_2 is the

sought displacement vector for both functions belonging to ϵ. For, it is a displacement vector for $f_2(x, y)$ belonging to $\epsilon/2$, and hence a fortiori for ϵ. And, as S_1, T_1 is a displacement vector for $f_1(x, y)$ belonging to $\epsilon/2$, and $|S_2-S_1|<\delta$, $|T_2-T_1|<\delta$, from the choice of δ it follows that S_2, T_2 is a displacement vector for $f_1(x, y)$ belonging to ϵ.

Corollary. *Any n almost periodic functions are simultaneously almost periodic, that is we may find an r such that in each circle of this radius there is a single displacement vector s, t which belongs to ϵ for all n functions.*

Theorem IV. *If $\phi(x_1, x_2, \ldots x_n)$ is continuous in the region $|x_i|<A_i$, and $u_1, u_2, \ldots u_n$ are n almost periodic functions of x and y such that $|u_i|<A_i'<A_i$, then $\phi(u_1, u_2, \ldots u_n)$ is an almost periodic function of x and y.*

For, in the closed region $|x_i|\leq A_i'$, $\phi(x_1, x_2, \ldots x_n)$ is uniformly continuous. Hence we may take δ such that $|\phi(x_1', x_2', \ldots x_n')-\phi(x_1, x_2, \ldots x_n)|<\epsilon$, provided $|x_i'-x_i|<\delta$. Also, let r be the displacement radius belonging to δ, holding simultaneously for all the n functions, which exists by the corollary to Theorem III. Then r is the displacement radius for $\phi(u_1, u_2, \ldots u_n)$ considered as a function of x and y. For, at two points differing by the common displacement vector, x, y, and $x+s$, $y+t$, each u_i differs by less than δ, and therefore at these two points ϕ differs by less than ϵ.

Corollary 1. *The sum of two, or of n almost periodic functions is almost periodic.*

Corollary 2. *The product of two, or of n almost periodic functions is almost periodic.*

Corollary 3. *The quotient of two almost periodic functions, provided the lower limit of the absolute value of the divisor is greater than zero, is almost periodic.*

Corollary 4. *The absolute value of any almost periodic function is almost periodic.*

Theorem V. *Any uniformly convergent series of almost periodic functions of two variables is an almost periodic function of two variables.*

For, take N so large that the remainder after n terms in the series is less than $\epsilon/3$. By Theorem IV, Corollary 1, the sum of the first N terms, S_N is an almost periodic function. Accordingly we may find a displacement radius r for it belonging to $\epsilon/3$. This will be a displacement radius for S, the sum of the series to infinity, belonging to ϵ. For, if s, t is the displacement vector for S_N belonging to $\epsilon/3$, we have:

$$|S_N(x+s, y+t)-S_N(x, y)| \leq \epsilon/3,$$

But:

$$|S(x, y)-S_N(x, y)| < \epsilon/3$$

and

$$|S(x+s, y+t)-S_N(x+s, y+t)| < \epsilon/3.$$

Consequently we have:

$$|S(x+s, y+t)-S(x, y)| < \epsilon.$$

Corollary 1. *A uniformly convergent series of continuous functions (continuous in a region including the upper bounds of the arguments) of almost periodic functions is almost periodic.*

Corollary 2. *If the series* $\sum_{i,j} A_{p_i q_j} e^{i p_i x + i q_j y}$ *converges uniformly, and hence in particular if* $\sum_{i,j} |A_{p_i q_j}|$ *converges, it represents an almost periodic function.*

We shall now prove some theorems which lead to the existence of constants analogous to the average value and Fourier coefficients for ordinary periodic functions.

Theorem VI. (*Mean value theorem.*) *If* $f(x, y)$ *is almost periodic,*

$$\lim_{T=\infty} 1/T^2 \int_0^T \int_0^T f(x, y)dxdy$$

exists.

We call it the *mean value* of $f(x, y)$, and abbreviate it by $M\{f(x, y)\}$.

We put $I(T)=1/T^2\int_0^T\int_0^T f(x, y)\ dxdy$, and shall show that $I(\infty)$ exists by showing that, for $T_1>T_\epsilon$, $T_2>T_\epsilon$.

$$|I(T_1)-I(T_2)|<\epsilon.$$

This will follow if we can find an X_ϵ and a T_ϵ such that:

$$|I(T_1)-I(X_\epsilon)\ |<\epsilon/2,$$

for $T_1>T_\epsilon$.

Let G be an upper bound of $f(x, y)$, shown to exist in Theorem I, and let a be a displacement side for $\eta=\epsilon/18$. We put $X=aG/\eta$ and $T_\epsilon>XG/\eta$, $X+a$. We must now prove the inequality above for these values.

Consider the region of integration of $I(T_\epsilon)$. It is a square of side $T_\epsilon=M(X+a)+\theta(X+a)$, where M is an integer and $|\ \theta\ |<1$. Hence we may divide it into M^2 squares of side $X+a$, with sides parallel to the axes, and a remainder area. In the square of side a at the lower left hand corner of one of these squares of side $X+a$, we may find a displacement s, t for $f(x, y)$ belonging to η. This vector will carry the square with sides $x=0$, $x=X$, $y=0$, $y=X$, on a square inside the one with side $X+a$. We have inside this latter square of side X:

$$\left|\int_s^{s+X}\int^{T+X} f(x, y)\ dxdy-\int_0^X\int_0^X f\ (x, y)\ dxdy\right|<X^2\eta$$

since

$$|f(x+s, y+t)-f(x, y)|\ <\eta.$$

The integral over the T_ϵ square is made up of several parts. Over the squares of side X. by the nequality just written, we have:

$$M^2\int_0^X\int_0^X (fx\ \ y)\ dxdy+kM^2X^2\eta,\ |k\ |<1.$$

Over the remaining area, in magnitude $T_\epsilon^2-M^2X^2$, since $|f(x, y)|<G$, we have:

$$k'(T_\epsilon^2-M^2X^2)G,\qquad |k'|<1.$$

Since $I(T_\epsilon)$ is the sum of these integrals divided by T_ϵ^2,

$$I\ (T_\epsilon)=M^2/T_\epsilon^2\int_0^X\int_0^X f(x, y)\ dxdy+\frac{kM^2X^2\eta+k'(T_\epsilon^2-M^2X^2)G}{T_\epsilon^2}.$$

We now notice that

$$1/X^2 \int_l^X \int^X f(x, y)\ dxdy - M^2/T_\epsilon^2 \int_0^X \int_0^X f(x, y)dxdy$$
$$= \frac{T_\epsilon^2 - M^2X^2}{X^2T_\epsilon^2} \int_0^X \int_0^X f(x, y)\ dxdy$$
$$= \frac{T_\epsilon^2 - M_\epsilon^2X^2}{X^2T_\epsilon^2} k''GX^2 = \frac{k''(T_\epsilon^2 - M^2X^2)G}{T_\epsilon^2}, \quad |k''| < 1.$$

Consequently,

$$I(X) = M^2/T^2 \int_0^X \int_0^X f(x, y)\ dxdy + \frac{k''(T_\epsilon^2 - M^2X^2)G}{T_\epsilon^2}.$$

Combining this with a previous equation:

$$I(T_\epsilon) - I(X) = \frac{kM^2X^2\eta + (k' + k'')(T_\epsilon^2 - M^2X^2)G}{T_\epsilon^2}.$$

But $MX < T_\epsilon$, $|k|$, $|k'|$, $|k''| < 1$, and

$$(T_\epsilon^2 - M^2X^2)G = (T_\epsilon + MX)(MaG + \theta aG + \theta GX)$$
$$< 2T_\epsilon(MX + \theta X + \theta T_\epsilon) < 4T_\epsilon^2\eta,$$

from which it follows that:

$$|I(T) - I(X)| < 9\eta = \epsilon/2,$$

the inequality we were seeking to prove.

Theorem VII. (*Sharper mean value theorem.*) *If $f(x, y)$ is almost periodic,*

$$\lim_{T=\infty} 1/T^2 \int_a^{a+T} \int_b^{b+T} f(x, y)\ dxdy$$
$$= \lim_{T=\infty} 1/T^2 \int_0^T \int_0^T f(x-a, y-b)dxdy$$

exists uniformly with respect to a and b, and equals $M\{f(x, y)\}$.

This follows at once from the proof of Theorem VI, which may be applied to the region of integration of this theorem, the uniformity resulting from the fact that the inequalities do not involve a or b. As the $I(X)$ is still a square with one corner at the origin, the limit here is again $M\{f(x, y)\}$.

Corollary 1. *In particular, the limits:*

$$\lim_{T=\infty} 1/T^2 \int_{-T}^{0}\int_{-T}^{0} f(x, y)\, dxdy \text{ and } \lim_{T=\infty} 1/4T^2 \int_{-T}^{T}\int_{-T}^{T} f(x, y)\, dxdy$$

exist and equal $M\{f(x, y)\}$.

Corollary 2. *Such limits as*

$$\lim_{T=\infty} 1/T^2 \int_{0}^{T}\int_{0}^{T} f(x+a_1, y+b_1)\, f(x+a_2, y+b_2)\, dxdy$$

exist uniformly with respect to a and b, and, more generally:

$$\lim_{T=\infty} \int_{0}^{T}\int_{0}^{T} \phi(f_i(x+a_i, y+b_i))\, dxdy,$$

where $\phi(f_i)$ *is continuous in the region* $|f_i| < G_i$.

This corollary follows from the fact that the T_ϵ used in proving Theorem VI depends on the a_i, ϵ and G, and on no other quantities.

Corollary 3. *If* $f(x, y)$ *is almost periodic, the limit:*

$$\lim_{S=\infty} 1/S \int_{S}\int f(x, y)\, dxdy$$

exists and equals $M\{f(x, y)\}$, *where* S *is any regular region of fixed shape.*

We carry through the proof as follows. First select T such that the $I(T)$ of Theorem VI, and hence that of Theorem VII is within $\epsilon/2$ of its limit $M\{f(x, y)\}$. Next take a region S_1 of unit area, and select η so small that if a network of squares of side η is placed on this S_1 with any orientation, and the N squares entirely in S_1 are disregarded, the remainder area, $S_1 - N\eta^2$, will be less than $\epsilon/2(|M\{f(x, y)\}|+G)$, where G is an upper bound for $|f(x, y)|$. Now take S_2 a region of area $(T/\eta)^2$, and consider the value of

$$1/S \int_{S}\int f(x, y)\, dxdy$$

for $S > S_2$.

On applying a network of squares of side T to S, there will be N' squares entirely in S, and the remainder area will be less than

$\epsilon S/2(|M\{f(x, y)\}|+G)$. The value of the integral over any square will be

$$M\{f(x, y)\}\, T^2+k\epsilon T^2/2, \quad |k|<1.$$

Over the remainder area the value will be

$$k'(S-N'T^2)G, \qquad |k'|<1.$$

Hence the average over the whole of S will be

$$\frac{M\{f(x, y)\}\, N'T^2+k\epsilon N'T^2/2+k'(S-N'T^2)G}{S}$$

or

$$M\{f(x, y)\}+N'k\epsilon T^2/2S+(S-N'T^2)(k'G-M\{f(x, y)\})/S.$$

But, since

$$NT^2 < S, \ |k|<1, \ |k'|<1$$

and

$$(S-N'T^2)\,|\,k'G-M\{f(x,y)\}\,|$$

$$< (S-N'T^2)\ (G+|M\{f(x,y)\}|)< \epsilon S/2$$

this average differs from $M\{f(x, y)\}$ by a quantity in absolute value less than ϵ, and hence approaches $M\{f(x,y)\}$ as a limit when S becomes infinitive.

We are now in a position to define quantities which play the same rôle for our almost periodic functions of two variables that the Fourier coefficients do for doubly periodic functions. We begin by noting that, by Corollary 2 to Theorem IV, the function $f(x, y)\ e^{-i\lambda x-i\mu y}$ is almost periodic. Hence, by Theorem VI, its mean value exists. We define

$$\alpha(\lambda, \mu)=M\{f(x, y)e^{-i\lambda x-i\mu y}\}$$

$$=\lim_{T=\infty} 1/T^2\int_0^T\int_0^T f(x, y)e^{-i\lambda x-i\mu}dxdy.$$

For later use, we note that:

$$M\{e^{i\lambda x+i\mu y}\}=\lim_{T=\infty} 1/T^2\int_0^T\int_0^T e^{i\lambda x+i\mu y}dxdy=\begin{cases}0, \ \lambda=\mu=0\\ 1, \ \lambda \text{ or } \mu\neq 0,\end{cases}$$

i.e., vanishes unless $\lambda=\mu=0$, in which case it equals unity. This is readily verified by direct calculation.

As a first property of the constants $\alpha(\lambda, \mu)$, we have·

Theorem VIII. *If $f(x, y)$ is an almost periodic function, and λ_m, μ_n are any real numbers, and b_{mn} any complex numbers, then:*

$$M\left\{ \left| f(x,y) - \sum_{m=1}^{M}\sum_{n=1}^{N} b_{mn}\, e^{i\lambda_m x + i\mu_n y} \right|^2 \right\} =$$

$$= M\left\{ |f(x,y)|^2 \right\} - \sum_{1}^{MN}\sum_{1}^{} |a(\lambda_m, \mu_n)|^2 + \sum_{1}^{MN}\sum_{1}^{} |b_{mn} - a(\lambda_m, \mu_n)|^2.$$

To prove this, we use $\bar{z}$ to denote the complex number conjugate to z, and write:

$$M\left\{ \left| f(x,y) - \sum_{1}^{MN}\sum_{1}^{} b_{mn}\, e^{i\lambda_m x + i\mu_n y} \right|^2 \right\}$$

$$= M\left\{ \left[f(x,y) - \sum_{1}^{M}\sum_{1}^{N} b_{mn} e^{i\lambda_m x + i\mu_n y} \right] \left[\bar{f}(x,y) - \sum_{1}^{M}\sum_{1}^{N} \bar{b}_{mn} e^{-i\lambda_m x - i\mu_n y} \right] \right\}$$

$$= M\left\{ f(x,y)\bar{f}(x,y) \right\} - \sum_{1}^{MN}\sum_{1}^{} \bar{b}_{mn} M\left\{ f(x,y) e^{-i\lambda_m x - i\mu_n y} \right\}$$

$$- \sum_{1}^{MN}\sum_{1}^{} b_{mn} M\left\{ \bar{f}(x,y) e^{i\lambda_m x + i\mu_n y} \right\}$$

$$+ \sum_{1}^{MN}\sum_{1}^{MN}\sum_{1}^{}\sum_{1}^{} b_{mn}\bar{b}_{m'n'} M\left\{ e^{i(\lambda_m - \lambda_{m'})x + i(\mu_n - \mu_{n'})y} \right\}.$$

By the average value property of the exponential terms, mentioned above, we see that the only terms of the last sum which do not vanish are those for which the exponents reduce to zero. Also, the expressions in the second and third terms respectively are the values of $a(\lambda_m, \mu_n)$ and its conjugate, by the definition of these quantities. Accordingly the above expression reduces to:

$$M\left\{ f(x,y)\bar{f}(x,y) \right\} - \sum_{1}^{MN}\sum_{1}^{} b_{mn}\, a(\lambda_m, \mu_n)$$

$$- \sum_{1}^{MN}\sum_{1}^{} b_{mn}\bar{a}(\lambda_m, \mu_n) + \sum_{1}^{MN}\sum_{1}^{} b_{mn}\bar{b}_{mn}$$

$$= M\left\{ |f(x,y)|^2 \right\} - \sum_{1}^{MN}\sum_{1}^{} a(\lambda_m, \mu_n)\ \bar{a}(\lambda_m, \mu_n)$$

$$+ \sum_{1}^{MN}\sum_{1}^{} \left[b_{mn} - a(\lambda_m, \mu_n) \right] \left[\bar{b}_{mn} - \bar{a}(\lambda_m, \mu_n) \right]$$

$$= M\left\{ |f(x,y)|^2 \right\} - \sum_{1}^{MN}\sum_{1}^{} |a(\lambda_m, \mu_n)|^2 + \sum_{1}^{MN}\sum_{1}^{} |b_{mn} - a(\lambda_m, \mu_n)|^2.$$

This proves the theorem.

The theorem we have just proved enables us to state an interesting result about approximations to our almost periodic function. If we think of the best approximation as that which has the average value of the square of the absolute value of the error over the infinite interval least, and seek the linear combination of MN exponential terms, $\sum_{1}^{M}\sum_{1}^{N} b_{mn}e^{i\lambda_m x+i\mu_n y}$, which best approximates our function, we see that it is the sum obtained by setting $b_{mn}=\alpha(\lambda_m, \mu_n)$. For this choice causes the last term in the right member of our equation to vanish, and consequently minimizes the left member, which is the average value of the absolute value of the error squared. We express the result of this substitution by

Theorem IX. *If $f(x, y)$ is an almost periodic function, and λ_m, μ_n are any real numbers, then:*

$$M\left\{|f(x, y)-\sum_{1}^{M}\sum_{1}^{N}\alpha(\lambda_m, \mu_n)e^{i\lambda_m x+i\mu_n y}|^2\right\}=M\left\{|f(x, y)|^2\right\}-\sum_{1}^{M}\sum_{1}^{N}|\alpha(\lambda_m, \mu_n)|^2.$$

Corollary. $\sum_{1}^{M}\sum_{1}^{N}|\alpha(\lambda_m, \mu_n)|^2 \leq M\left\{|f(x, y)|^2\right\}$

For the left member of the equation of Theorem IX is essentially positive, or zero.

Theorem X. *There are only an enumerable number of pairs λ_m, μ_n, for which $\alpha(\lambda_m, \mu_n)$ is different from zero.*

For, by the corollary to Theorem IX,

$$\sum_{1}^{M}\sum_{1}^{N}|\alpha(\lambda_m, \mu_n)|^2 \leq M\left\{|f(x, y)|^2\right\}$$

and consequently is bounded, for all values of M, N. Hence the number of pairs λ_m, μ_n such that $|\alpha(\lambda_m, \mu_n)|^2$ exceeds $1/n$ must be finite. Taking first those pairs for which the number $|\alpha(\lambda_m, \mu_n)|^2$ exceeds 1, then those for which it is less than or equal to 1, but greater than or equal to $\frac{1}{2}$, and so on, we see we may arrange the

pairs for which $\alpha(\lambda_m, \mu_n)$ is not zero in order, and hence their number is enumerable.

This theorem enables us to get an infinite series of terms for an almost periodic function, its generalized Fourier series, analogous to the Fourier series for a doubly periodic function, composed of all the terms $\alpha(\lambda_m, \mu_n)\, e^{i\lambda_m x + i\mu_n y}$ for which $\alpha(\lambda_m, \mu_n)$ does not vanish. We have here no natural method of ordering the terms, according to the exponents, since taking the exponents in order may not give an enumerable arrangement. By the proof of Theorem X, we may arrange them in order of decreasing $\alpha(\lambda_m, \mu_n)$ however. Our next theorem shows that, if our almost periodic is actually doubly periodic, the series here obtained is the classical Fourier series for the function.

Theorem XI. *If $f(x, y)$ is doubly periodic, the x period being $p = 2\pi/K$, and the y period being $q = 2\pi/L$, the series* $\sum_1^\infty \sum_1^\infty \alpha(\lambda_m, \mu_n)\, e^{i\lambda_m x + i\mu_n y}$ *agrees with the ordinary Fourier series* $\sum_1^\infty \sum_1^\infty a_{mn} e^{imKx + inLy}$, *provided that in the latter series terms with zero coefficients are suppressed.*

To prove this, we must show that for $\lambda_m = mK$ and $\mu_n = nL$, $\alpha(\lambda_m, \mu_n) = 1/pq \int_0^p \int_0^q f(x, y)\, e^{-imKx - inLy} dxdy = a_{mn}$, *and in other cases equals zero.*

We have by definition

$$\alpha(mK, nL) = \lim_{T=\infty} 1/T^2 \int_0^T \int_0^T f(x, y)\, e^{-imKx - inLy}\, dxdy.$$

By Corollary 3 to Theorem VII, this limit is equal to that for a rectangle the ratio of whose sides is p/q, *i.e.*, it equals

$$\lim_{T=\infty} 1/T^2 pq \int_0^{pT} \int_0^{qT} f(x, y)\, e^{-imKx - inLy}\, dxdy.$$

Since this limit exists, its value may be computed by taking the limit as T becomes infinite through integral values. For such values, because of the periodicity of $f(x, y)$, it is constantly equal to

$$1/pq \int_0^p \int_0^q f(x, y)\, e^{imx + iny}\, dxdy = a_{mn},$$

which accordingly is the value of the limit.

To show that all other coefficients $a(\lambda_m, \mu_n)$ vanish, we approximate $f(x, y)$ by a trigonometric polynomial,

$$f(x, y) = \sum_{1}^{M}\sum_{1}^{N} b_{mn} e^{imx+iny} + R(x, y)$$

in such a way that the error is in absolute value less than ϵ.

$$|R(x, y)| < \epsilon.$$

We may do this, since $f(x, y)$ being periodic, and continuous, is a continuous function of $X = e^{iKx}$, $Y = e^{iLy}$. Accordingly, by the Weierstrass approximation theorem[3], it may be approximated by a polynomial in X and Y to any uniform degree of approximation. This polynomial, in X and Y is, in terms of x and y, the sum $\sum_{1}^{M}\sum_{1}^{N} b_{mn} e^{imx+iny}$ here required.

We then have, for any values of λ_m, μ_n distinct from mK, nL.

$$a(\lambda_m, \mu_n) = M\{f(x, y)\, e^{-i\lambda_m x - i\mu_n y}\}$$

$$= M\left\{\sum_{1}^{M}\sum_{1}^{N} b_{mn} e^{-i(mK-\lambda_m)x + i(nL-\mu_n)y}\right\} + M\left\{R(x, y)\, e^{-i\lambda_m x - i\mu_n y}\right\}$$

$$= M\{R(x, y)\, e^{-i\lambda_m x - i\mu_n y}\}$$

by the average value property of the exponential terms, and our hypothesis on λ_m, μ_n. But since the exponential factor of $R(x, y)$ is one in absolute value, and $R(x, y)$ is in absolute value less than ϵ, we see that:

$$|a(\lambda_m, \mu_n)| < \epsilon,$$

and consequently:

$$a(\lambda_m, \mu_n) = 0,$$

which completes the proof of the theorem.

Theorem XII. *If the series* $\sum_{1}^{\infty}\sum_{1}^{\infty} a_{mn} e^{i\lambda_m x + i\mu_n y}$, *the exponents being all different and the coefficients distinct from zero, converges uniformly in the entire infinite plane, or has only a finite number of terms, it represents an almost periodic function, whose generalized Fourier series coincides with the given series.*

[3] Weierstrass, Berliner Sitzungsberichte, 1885. For an elementary proof for the case of two variables, compare this Journal, Vol. IV., p. 148.

That the series represents an almost periodic function has already been proved in Corollary 2, Theorem V. To prove the rest of the statement, we note that, since the series converges uniformly, integration, and hence the operation of taking the mean, may be performed termwise. This remains true when factors of absolute value unity, like $e^{-i\lambda x - i\mu y}$ are multipled in, since these do not destroy the uniform convergence. From this we see that $\alpha(\lambda_m, \mu_n) = a_{mn}$ from the average value property of the exponential terms.

Corollary. *There is no restriction on the exponents which may arise for a given almost periodic function.*

For they may be chosen as any enumerable set, *e.g.*, all pairs of rational numbers, and the coefficients then chosen, *e.g.*, so that $\sum_1^\infty \sum_1^\infty |a_{mn}|$ converges, so that the series $\Sigma\Sigma a_{mn} e^{i\lambda_m x + i\mu_n x}$ converges uniformly.

We note that there is undoubtedly a restriction on the coefficients, since, for example, the coefficients and exponents of the ordinary Fourier series for a discontinuous doubly periodic function probably correspond to no almost periodic function in our sense.

Theorem XIII. *If the number of non-vanishing coefficients of an almost periodic function is infinite, and we denote them by* $A_{mn} = \alpha(\lambda_m, \mu_n)$ *then* $\sum_1^\infty \sum_1^\infty |A_{mn}|^2$ *converges, and we have:*

$$\sum_1^\infty \sum_1^\infty |A_{mn}|^2 \leq M \left\{ |f(x, y)|^2 \right\}.$$

This follows from the corollary to Theorem IX. It is the analogue of Bessel's inequality in the ordinary trigonometric theory.

That the analogue of Parseval's theorem holds, that is, that in the above relation we have equality in all cases shall not be proved here. We postpone the discussion of this, the "fundamental theorem" for almost periodic functions of two variables, to a later paper.

ADJOINT AND INVERSE DETERMINANTS AND MATRICES

By Lepine Hall Rice

The law of transformation of a tensor suggests a study of the determinants and matrices involved in more general transformations. Let us first consider the matrix of the coefficients in the n^2 equations each with n^2 terms on the right:

$$Y_{ij} = \Sigma_{rs} a_{ir} b_{js} X_{rs}, \tag{1}$$

where the X's are any n^2 quantities, and the a's and b's are quantities subject only to the restriction

$$A = |a_{ir}| \neq 0, \quad B = |b_{js}| \neq 0. \tag{2}$$

We denote the matrix by

$$|| (a_{ir} b_{js})_{(ij)\,(rs)} ||, \tag{3}$$

and say that the locant $(ij)(rs)$ of the general element $a_{ir}b_{js}$ is a double bipartite index, (ij) and (rs) being single bipartite indices which take the values $11, 12, \ldots, 1n, 21, 22 \ldots, 2n, \ldots, n1, n2, \ldots, nn$. The matrix of the third order written in full is:

$$\left\| \begin{array}{ccccccccc}
a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & a_{12}b_{11} & a_{12}b_{12} & a_{12}b_{13} & a_{13}b_{11} & a_{13}b_{12} & a_{13}b_{13} \\
a_{11}b_{21} & a_{11}b_{22} & a_{11}b_{23} & a_{12}b_{21} & a_{12}b_{22} & a_{12}b_{23} & a_{13}b_{21} & a_{13}b_{22} & a_{13}b_{23} \\
a_{11}b_{31} & a_{11}b_{32} & a_{11}b_{33} & a_{12}b_{31} & a_{12}b_{32} & a_{12}b_{33} & a_{13}b_{31} & a_{13}b_{32} & a_{13}b_{33} \\
a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} & a_{22}b_{11} & a_{22}b_{12} & a_{22}b_{13} & a_{23}b_{11} & a_{23}b_{12} & a_{23}b_{13} \\
a_{21}b_{21} & a_{21}b_{22} & a_{21}b_{23} & a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} & a_{23}b_{21} & a_{23}b_{22} & a_{23}b_{23} \\
a_{21}b_{31} & a_{21}b_{32} & a_{21}b_{33} & a_{22}b_{31} & a_{22}b_{32} & a_{22}b_{33} & a_{23}b_{31} & a_{23}b_{32} & a_{23}b_{33} \\
a_{31}b_{11} & a_{31}b_{12} & a_{31}b_{13} & a_{32}b_{11} & a_{32}b_{12} & a_{32}b_{13} & a_{33}b_{11} & a_{33}b_{12} & a_{33}b_{13} \\
a_{31}b_{21} & a_{31}b_{22} & a_{31}b_{23} & a_{32}b_{21} & a_{32}b_{22} & a_{32}b_{23} & a_{33}b_{21} & a_{33}b_{22} & a_{33}b_{23} \\
a_{31}b_{31} & a_{31}b_{32} & a_{31}b_{33} & a_{32}b_{31} & a_{32}b_{32} & a_{32}b_{33} & a_{33}b_{31} & a_{33}b_{32} & a_{33}b_{33}
\end{array} \right\| \tag{4}$$

Direct product.

The inverse of the matrix (4) is easily written. Denoting by A_{ir} and B_{js} as usual the cofactors of a_{ir} and b_{js} in A and B respectively, and by a^{ir} and b^{js} the quotients A_{ir}/A and B_{js}/B, and calling a^{ir} and b^{js} the *reciprocals* of a_{ir} and b_{js}, we replace every a

and every b in (4) by its reciprocal, and then transpose rows and columns. The (ij)th row of (4) is

$$a_{i1}b_{j1}\ a_{i1}b_{j2}\ a_{i1}b_{j3}\ a_{i2}b_{j1}\ a_{i2}b_{j2}\ a_{i2}b_{j3}\ a_{i3}b_{j1}\ a_{i3}b_{j2}\ a_{i3}b_{j3} \tag{5}$$

while the $(i'j')$th column of the new matrix is

$$a^{i'1}b^{j'1}\ a^{i'1}b^{j'2}\ a^{i'1}b^{j'3}\ a^{i'2}b^{j'1}\ a^{i'2}b^{j'2}\ a^{i'2}b^{j'3}\ a^{i'3}b^{j'1}\ a^{i'3}b^{j'2}\ a^{i'3}b^{j'3} \tag{6}$$

and the product may be written

$$\begin{aligned} a_{i1}a^{i'1}(b_{j1}b^{j'1}+b_{j2}b^{j'2}+b_{j3}b^{j'3})+a_{i2}a^{i'2}(b_{j1}b^{j'1}+\dots) \\ +a_{i3}a^{i'3}(b_{j1}b^{j'1}+\dots) \end{aligned} \tag{7}$$

which vanishes unless $j'=j$ and then reduces to

$$a_{i1}a^{i'1}+a_{i2}a^{i'2}+a_{i3}a^{i'3},$$

which in turn vanishes unless $i'=i$ and then equals unity. The product of the two matrices is therefore the unit matrix and the new matrix is the inverse of the old. As the argument holds whatever the order, we have proved:

Theorem 1. The matrix the inverse of (3), *i.e.*, of

$$\left|\left| (a_{ir}b_{js})_{(ij)\ (rs)} \right|\right|, \tag{8}$$

is

$$\left|\left| (a^{ir}b^{js})_{(rs)\ (ij)} \right|\right|, \tag{9}$$

that is, it may be obtained by replacing each a and each b with its reciprocal and then transposing the rows and columns.

Passing from the case of two to that of three determinants

$$A=|a_{ir}|\neq 0, \qquad B=|b_{js}|\neq 0, \qquad C=|c_{kt}|\neq 0, \tag{10}$$

our matrix will be

$$\left|\left| (a_{ir}b_{js}c_{kt})_{(ijk)\ (rst)} \right|\right|, \tag{11}$$

of order n^3, each tripartite index taking the values 111, 112, . . . , $11n$, 121, 122, . . . , $12n$, . . . , $1n1$, $1n2$, . . . , $1nn$, . . . , $nn1$, $nn2$, . . . , nnn. And the inverse matrix will be

$$\left|\left| (a^{ir}b^{js}c^{kt})_{(rst)\ (ijk)} \right|\right|; \tag{12}$$

for on multiplying the (ijk)th row of (11) and the $(i'j'k')$th column of (12) we shall have

$$\Sigma_{rst}a_{ir}b_{js}c_{kt}a^{i'r}b^{j's}c^{k't}=\Sigma_r a_{ir}a^{i'r}\Sigma_s b_{js}b^{j's}\Sigma_t c_{kt}c^{k't}, \tag{13}$$

which vanishes unless $k'=k$, $j'=j$, and $i'=i$, and then equals unity. As the argument holds whatever the number of determinants, we have proved with respect to the q determinants

$$A=|a_{ir}|\neq 0, \quad B=|b_{js}|\neq 0, \quad \ldots, E=|e_{mv}|\neq 0, \tag{14}$$

and the resulting matrix of order n^q, the following theorem:

Theorem 2. The matrix the inverse of

$$||\,(a_{ir}b_{js}\ldots e_{mv})_{(ij\ldots m)\,(rs\ldots v)}\,|| \tag{15}$$

is

$$||\,(a^{ir}b^{js}\ldots e^{mv})_{(rs\ldots v)\,(ij\ldots m)}\,||\,, \tag{16}$$

that is, it may be obtained by replacing each constituent of each element with its reciprocal and then transposing the rows and columns.

A different sort of generalization of the matrix (3) is found in the matrix

$$||\,(a_{ijr}b_{jrs})_{(ij)\,(rs)}\,||\,, \tag{17}$$

in which n^3 a's and n^3 b's are involved. The a's and the b's themselves fall into 3-way matrices

$$||\,a_{ijr}\,||\,, \qquad ||\,b_{jrs}\,||\,. \tag{18}$$

For the case $n=3$ we shall exhibit the matrices (17) and (18), using the notation

$$\underset{ijr}{a}=a_{ijr}, \qquad \underset{jrs}{b}=b_{jrs}, \qquad \underset{ijrs}{ab}=a_{ijr}b_{jrs}: \tag{19}$$

$$\left\|\begin{array}{ccc|ccc|ccc}
a_{111} & a_{112} & a_{113} & a_{121} & a_{122} & a_{123} & a_{131} & a_{132} & a_{133} \\
a_{211} & a_{212} & a_{213} & a_{221} & a_{222} & a_{223} & a_{231} & a_{232} & a_{233} \\
a_{311} & a_{312} & a_{313} & a_{321} & a_{322} & a_{323} & a_{331} & a_{332} & a_{333}
\end{array}\right\|, \tag{20}$$

$$\left\|\begin{array}{ccc|ccc|ccc}
b_{111} & b_{112} & b_{113} & b_{121} & b_{122} & b_{123} & b_{131} & b_{132} & b_{133} \\
b_{211} & b_{212} & b_{213} & b_{221} & b_{222} & b_{223} & b_{231} & b_{232} & b_{233} \\
b_{311} & b_{312} & b_{313} & b_{321} & b_{322} & b_{323} & b_{331} & b_{332} & b_{333}
\end{array}\right\|, \tag{21}$$

$$\left\| \begin{array}{ccccccccc}
ab & ab & ab & ab & ab & ab & ab & ab & ab \\
1111 & 1112 & 1113 & 1121 & 1122 & 1123 & 1131 & 1132 & 1133 \\
ab & ab & ab & ab & ab & ab & ab & ab & ab \\
1211 & 1212 & 1213 & 1221 & 1222 & 1223 & 1231 & 1232 & 1233 \\
ab & ab & ab & ab & ab & ab & ab & ab & ab \\
1311 & 1312 & 1313 & 1321 & 1322 & 1323 & 1331 & 1332 & 1333 \\
ab & ab & ab & ab & ab & ab & ab & ab & ab \\
2111 & 2112 & 2113 & 2121 & 2122 & 2123 & 2131 & 2132 & 2133 \\
ab & ab & ab & ab & ab & ab & ab & ab & ab \\
2211 & 2212 & 2213 & 2221 & 2222 & 2223 & 2231 & 2232 & 2233 \\
ab & ab & ab & ab & ab & ab & ab & ab & ab \\
2311 & 2312 & 2313 & 2321 & 2322 & 2323 & 2331 & 2332 & 2333 \\
ab & ab & ab & ab & ab & ab & ab & ab & ab \\
3111 & 3112 & 3113 & 3121 & 3122 & 3123 & 3131 & 3132 & 3133 \\
ab & ab & ab & ab & ab & ab & ab & ab & ab \\
3211 & 3212 & 3213 & 3221 & 3222 & 3223 & 3231 & 3232 & 3233 \\
ab & ab & ab & ab & ab & ab & ab & ab & ab \\
3311 & 3312 & 3313 & 3321 & 3322 & 3323 & 3331 & 3332 & 3333
\end{array} \right\| . \qquad (22)$$

In (20) and (21) the second-way layers of the a-matrix and b-matrix are displayed. Each of these layers is the matrix of a 2-way determinant, and it is necessary to assume that these determinants do not vanish:

$$\begin{aligned} |(a_{ijr})_{ir}| &\neq 0, \qquad j = 1, 2, \ldots, n. \\ |(b_{jrs})_{js}| &\neq 0, \qquad r = 1, 2, \ldots, n. \end{aligned} \qquad (23)$$

Let the cofactor of a_{ijr} divided by the determinant, *i.e.*, the reciprocal of a_{ijr}, be denoted by a^{ijr}_{--}, and the reciprocal of b_{jrs} by b^{jrs}_{--}. We shall prove

Theorem 3. The matrix the inverse of (17), *i.e.*, of

$$\| (a_{ijr} b_{jrs})_{(ij)\,(rs)} \| , \qquad (24)$$

is

$$\| (a^{ijr}_{--} b^{jrs}_{--})_{(rs)\,(ij)} \| , \qquad (25)$$

that is, it may be obtained by replacing each constituent of each element with its reciprocal in respect to that second-way layer in its own matrix in which it lies, and then transposing the rows and columns.

For, the product of the (ij)th row of (24) and the $(i'j')$th column of (25) is

$$\Sigma_{rs} a_{ijr} b_{jrs} \underline{\underline{a}}^{i'j'r} \underline{\underline{b}}^{j'rs} = \Sigma_r a_{ijr} \underline{\underline{a}}^{i'j'r} \Sigma_s b_{jrs} \underline{\underline{b}}^{j'rs}, \tag{26}$$

which vanishes unless $j'=j$, and then reduces to the sum with respect to r (with $j'=j$), which latter vanishes unless $i'=i$, and then equals unity.

From the matrix (17) we pass to the matrix

$$\| (a_{ijkr} b_{jkrs} c_{krst})_{(ijk)\,(rst)} \|, \tag{27}$$

of order n^3, whose elements are formed from elements of the 4-way matrices of order n

$$\| a_{ijkr} \|, \qquad \| b_{jkrs} \|, \qquad \| c_{krst} \|. \tag{28}$$

Each 4-way matrix may obviously be partitioned into n^2 2-way matrices of order n, in each of which the n^2 elements have like second-way indices and like third-way indices. We shall assume that the determinants of these 2-way matrices do not vanish, *i.e.*,

$$\begin{aligned} &|(a_{ijkr})_{ir}| \neq 0, && jk = 11, 12, \ldots, nn, \\ &|(b_{jkrs})_{js}| \neq 0, && kr = 11, 12, \ldots, nn, \\ &|(c_{krst})_{kt}| \neq 0, && rs = 11, 12, \ldots, nn. \end{aligned} \tag{29}$$

Conforming to our previous notation, we denote by $\underline{\underline{a}}^{ijkr}$ the reciprocal of a_{ijkr} with respect to $|(a_{ijkr})_{ir}|$, etc. And we shall prove

Theorem 4. The matrix the inverse of (27), *i.e.*, of

$$\| (a_{ijkr} b_{jkrs} c_{krst})_{(ijk)\,(rst)} \|, \tag{30}$$

is

$$\| (\underline{\underline{a}}^{ijkr} \underline{\underline{b}}^{jkrs} \underline{\underline{c}}^{krst})_{(rst)\,(ijk)} \|. \tag{31}$$

For, the product of the (ijk)th row of (30) and the $(i'j'k')$th column of (31) is

$$\Sigma_{rst} a_{ijkr} b_{jkrs} c_{krst} \underline{\underline{a}}^{i'j'k'r} \underline{\underline{b}}^{j'k'rs} \underline{\underline{c}}^{k'rst} = \Sigma_r a_{ijkr} \underline{\underline{a}}^{i'j'k'r} \Sigma_s b_{jkrs} \underline{\underline{b}}^{j'k'rs} \Sigma_t c_{krst} \underline{\underline{c}}^{k'rst} \tag{32}$$

which vanishes unless $i'j'k'=ijk$, and then equals unity.

From three we pass to q constituents drawn from the q $(q+1)$-way matrices

$$\| a_{ij\ldots mr} \|, \quad \| b_{jk\ldots rs} \|, \ldots, \quad \| e_{mr\ldots uv} \|, \tag{33}$$

where qn^{q-1} 2-way determinants do not vanish, *viz.*,

$$|(a_{ij\ldots mr})_{ir}| \neq 0, \quad |(b_{jk\ldots rs})_{js}| \neq 0, \ldots, \quad |(e_{mr\ldots uv})_{mv}| \neq 0. \tag{34}$$

The generality of (32) is so clear that without further proof we give

Theorem 5. The matrix the inverse of

$$\left|\left|\,(a_{ij\ldots mr}b_{jk\ldots rs}\ldots e_{mr\ldots uv})_{(ij\ldots lm)\,(rs\ldots uv)}\,\right|\right| \tag{35}$$

is

$$\left|\left|\,(a^{ij\ldots mr}b^{jk\ldots rs}\ldots e^{mr\ldots uv})_{(rs\ldots uv)\,(ij\ldots lm)}\,\right|\right|, \tag{36}$$

that is, it may be obtained by replacing each constituent of each element with its reciprocal with respect to (34), and then transposing the rows and columns.

The 2-way matrices of the determinants in (34) may be alike in each of the q matrices in (33), in which case Theorem 5 reduces to Theorem 2.

The determinant of the matrix (35) is known, from a theorem of W. H. Metzler,[1] to be equal to the product of the determinants of (34), *i.e.*,

$$\left|(a_{ij\ldots mr}b_{jk\ldots rs}\ldots e_{mr\ldots uv})_{(ij\ldots lm)\,(rs\ldots uv)}\right| = \prod_{1}^{n^{q-1}}{}_{j\ldots m}\left|(a_{ij\ldots mr})_{ir}\right|\ldots\prod_{1}^{n^{q-1}}{}_{r\ldots u}\left|(e_{mr\ldots uv})_{mv}\right|. \tag{37}$$

For the case of Theorem 2 this result reduces to

$$\left|(a_{ir}b_{js}\ldots e_{mv})_{(ij\ldots m)\,(rs\ldots v)}\right| = (AB\ldots E)^{n^{q-1}}. \tag{38}$$

These 2-way matrices have a $2q$-way aspect, and the associated $2q$-way determinants are of interest. We shall begin by stating and proving some theorems about adjoint determinants of higher class. By the adjoint is meant, as usual, a determinant obtained by replacing each element of the given determinant with its cofactor, just as in the special case of 2-way determinants. The given determinant in our theorems is any determinant formed as the product of two or more determinants by Scott's law of multiplication as extended to determinants of any signancy (permanents included).[2] Thus the Scott product of two determinants A and B of classes p_1 and p_2 is, with proper signancy,

$$S'' = \left|(a_{\alpha_1\ldots\alpha_{p_1-1}\lambda}b_{\lambda\beta_1\ldots\beta_{p_2}})_{\alpha_1\ldots\alpha_{p_1-1}\lambda\beta_1\ldots\beta_{p_2}}\right|, \tag{39}$$

[1] *On a determinant each of whose elements is the product of k factors*, W. H. Metzler, Am. Math. Monthly, vol. VII, p. 151 (June–July, 1900). *A fundamental remark concerning determinantal notations with the evaluation of an important determinant of special form*, E. H. Moore, Am. Journal of Math., vol. 1–2, ser. 2, p. 177 (1900).

[2] *P-way determinants*, L. H. Rice, Am. Journal of Math., vol. XL, p. 242 (1918), see p. 249.

which is general since in the determinants the order of the indices is arbitrary and therefore the linked indices may be α_{p_1} and β_1. One form of Scott product of three determinants is

$$S''' = \left|(a_{\alpha_1\ldots\alpha_{p_1-1}\lambda} b_{\lambda\beta_2\ldots\beta_{p_2-1}\mu} c_{\mu\gamma_2\ldots\gamma_{p_3}})_{\alpha_1\ldots\alpha_{p_1-1}\lambda\beta_2\ldots\beta_{p_2-1}\mu\gamma_2\ldots\gamma_{p_3}}\right| \tag{40}$$

The corresponding form $S^{(q)}$ in the case of q determinants is manifest. In each case the signancy of the unlinked indices is continued from the factor determinants to the product, while the signancy of the linked indices is virtually continued in consequence of the rule that λ, μ, etc., shall be signant in the product if signant in one factor, but nonsignant in the product if signant in both or nonsignant in both factors. The result is that the sign of the cofactor of an element in $S^{(q)}$ is the product of the signs of the cofactors of its constituents (the a-element, the b-element, etc.) in A, B, etc. Not only so, but the cofactor aside from sign, that is, the minor complementary to the element in $S^{(q)}$, is a Scott product of the complementary minors in A, B, etc.; for complementary minors are determined by striking out one (the same in $S^{(q)}$ and the factor determinant) value from the range of each index. Thus the cofactor of an element in $S^{(q)}$ is the product of the cofactors of the constituents of the element. *E.g.*, in S'',

$$\operatorname{cof}(a_{\alpha_1\ldots\alpha_{p_1-1}\lambda} b_{\lambda\beta_2\ldots\beta_{p_2}}) = A_{\alpha_1\ldots\alpha_{p_1-1}\lambda} B_{\lambda\beta_2\ldots\beta_{p_2}}. \tag{41}$$

These things are true with respect to any Scott product, and we have the theorem and corollary:

Theorem 6. The adjoint of a Scott product determinant is the Scott product of the adjoints of the factor determinants.

Corollary. The adjoint of a Scott product of 2-way determinants is equal to the $(n-1)$th power of the Scott product.

Furthermore, any minor M'_S of order m in the adjoint S' of a Scott product S is the product of the corresponding minors M'_A, M'_B, etc., in the adjoints A', B', etc. Our meaning is illustrated by the case:

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \qquad B = \begin{vmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{vmatrix} \tag{42}$$

$$S = \begin{vmatrix} ab_{111} & ab_{112} & ab_{113} & ab_{114} & ab_{121} & ab_{122} & ab_{123} & ab_{124} & ab_{131} & ab_{132} & ab_{133} & ab_{134} & ab_{141} & ab_{142} & ab_{143} & ab_{144} \\ ab_{211} & ab_{212} & ab_{213} & ab_{214} & ab_{221} & ab_{222} & ab_{223} & ab_{224} & ab_{231} & ab_{232} & ab_{233} & ab_{234} & ab_{241} & ab_{242} & ab_{243} & ab_{244} \\ ab_{311} & ab_{312} & ab_{313} & ab_{314} & ab_{321} & ab_{322} & ab_{323} & ab_{324} & ab_{331} & ab_{332} & ab_{333} & ab_{334} & ab_{341} & ab_{342} & ab_{343} & ab_{344} \\ ab_{411} & ab_{412} & ab_{413} & ab_{414} & ab_{421} & ab_{422} & ab_{423} & ab_{424} & ab_{431} & ab_{432} & ab_{433} & ab_{434} & ab_{441} & ab_{442} & ab_{443} & ab_{444} \end{vmatrix}. \tag{41}$$

By definition, A' and B' are formed by replacing a's with A's and b's with B's; and by (41) S' may be formed by the same substitution. Take for the minor M_S'

$$\begin{vmatrix} AB_{111} & AB_{114} & AB_{131} & AB_{134} \\ AB_{311} & AB_{314} & AB_{331} & AB_{334} \end{vmatrix}; \tag{44}$$

the corresponding minors of A' and B' of which this minor is the product are

$$\begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix}, \quad \begin{vmatrix} B_{11} & B_{14} \\ B_{31} & B_{34} \end{vmatrix}. \tag{45}$$

that is, they are the minors in which the same indices (α_1 and α_1, α_2 and λ, β_1 and λ, β_2 and β_2) take the same m values in their ranges.

Now it is well known that if D' is the adjoint of any 2-way determinant D, and M and M' are corresponding m-rowed minors of D and D' respectively, then M' is equal to the product of D^{m-1} by the cofactor of M. By means of this theorem we can prove the following extension:

Theorem 7. Any minor of order m of the adjoint of a Scott product of 2-way determinants is equal to the cofactor of the corresponding minor of the Scott product, multiplied by the $(m-1)$th power of the Scott product.

For as this minor M_S' of S' is the product of the corresponding minors M_A', M_B', etc. in the adjoints A', B', etc. of the 2-way determinants, and each of these is equal to the cofactor of the

corresponding minor M_A, M_B, etc. in the 2-way determinant A, B, etc. multiplied by A^{m-1}, B^{m-1}, etc., we have

$$M'_S = M'_A M'_B \ldots = (AB \ldots)^{m-1} \operatorname{cof} M_A \operatorname{cof} M_B \ldots = S^{m-1} \operatorname{cof} M_S, \tag{46}$$

it being evident that the product of the signs of the cofactors of M_A, M_B, etc. is equal to the sign of the cofactor of M_S. *E.g.*, in (42)–(45) the a-cofactor and b-cofactor are

$$+\begin{vmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{vmatrix}, \qquad -\begin{vmatrix} b_{23} & b_{23} \\ b_{43} & b_{43} \end{vmatrix}, \tag{47}$$

the + sign of the a-cofactor being obtained as the product of the two − signs of the permutations 1324, 1324; and the − sign of the b-cofactor being the product (−) (+) of the signs of the permutations 1324, 1423; while the sign of cof M_S is −, it being the product (−) (+) of the first (α_1) and last (β_2) signs. Clearly the equality will hold under all conditions of signancy.

By a specialization of $S^{(q)}$ (see after (40)) we can obtain a form of product determinant which has not the appearance of a Scott product but to which Theorem 6 applies. First take (40), put $p_1 = p_2 = 3$ and $p_3 = 2$, and let A and B be determinants whose third-way layers are nonsignant and alike, while C is a determinant, *i.e.*,

$$A = |a^{\pm\pm\mp}_{\alpha_1\alpha_2\alpha_3}|, \qquad B = |b^{\pm\pm\mp}_{\beta_1\beta_2\beta_3}|, \qquad C = |c^{\pm\pm}_{\gamma_1\gamma_2}|; \tag{48}$$

we get the 6-way full-sign determinant

$$\bar{S}''' = |(a_{\alpha_1\alpha_2} b_{\beta_1\beta_2} c_{\gamma_1\gamma_2})^{\pm\pm\pm\pm\pm\pm}_{\alpha_1\alpha_2\beta_1\beta_2\gamma_1\gamma_2}|. \tag{49}$$

Any minor of order m in the adjoint of a 3-way determinant of order n with nonsignant layers alike is known[3] to be equal to the cofactor of the corresponding minor in the determinant itself, multiplied by the $(m-1)$th power of the determinant and by the numerical factor.

$$\frac{n!\ m!}{n^m (n-m)!}; \tag{50}$$

[3] *Abrégé de la Théorie des Déterminants à n Dimensions*, M. Lecat, Gand, 1911, p. 121.

from which, the adjoint itself is equal to the $(n-1)$th power of the determinant multiplied by the numerical factor $(n!)^2/n^n$. We are thus enabled to write

$$\text{adj } \overline{\overline{S}}''' = [(n!)^2/n^n]^2(\overline{\overline{S}}''')^{n-1}, \tag{51}$$

and more generally, for q factors, to give

Theorem 8. If a $2q$-way full-sign determinant T of order n is formed by taking for each element the product of q elements one from each of q 2-way matrices of order n, their locants forming the locant of the product, the adjoint of this determinant is equal to the $(n-1)$th power of the determinant multiplied by a numerical factor, *viz.*,

$$\text{adj } T = \left[\frac{(n!)^q}{n^n}\right]^{q-1} T^{n-1}. \tag{52}$$

By a proof like that of Theorem 7, we also obtain

Theorem 9. Any minor of order m of the adjoint of T is equal to the cofactor of the corresponding minor of T, multiplied by T^{m-1} and by k^{q-1}, k denoting the numerical factor given in (50).

It is intended to return to these matters in a subsequent paper

THE THEORY OF OVERVOLTAGE

By Max Knobel

In a previous article[1] some approximate values of the concentration of adsorbed gas on polarized electrodes were determined. A commutator method was used to measure the rate of increase of overvoltage with time and from the form of these curves, the concentrations were calculated. As an outgrowth of this work and a series of experimental papers[2] on overvoltage, the details of what is believed to be a satisfactory theory of overvoltage have been developed. The essential idea, namely, that the overvoltage is due to an excess concentration of monatomic gas, is not new, but up to the present time the mechanism of the action, and the application to *all* the factors on which overvoltage depends has not been clearly explained.

In making the above mentioned concentration calculations the assumptions were made (1) that the electromotively active gas is monatomic and (2) that the relation between surface concentration and pressure is given by an equation of the form of Freundlich's, namely

$$P = \text{const } H_a^m \tag{1}$$

where P and H_a represent respectively the pressure and surface concentration of say monatomic hydrogen, and m is constant and larger than unity.[3]

From the general equation for overvoltage

$$E = \frac{RT}{nF} ln \frac{P}{P'} \tag{2}$$

where P and P' are the pressures under overvoltage conditions

[1] Knobel, Jour. Am. Chem. Soc. 46, 2613 (1924).

[2] Current density effect, Knobel, Caplan and Eiseman, Trans. Am. Electrochem. Soc. **43** 55, (1923) Temperature effect, Knobel and Joy, *ibid.* **44** (1923), Pressure effect, Knobel Jour. Am. Chem. Soc. **46** 2751 (1924) Surface Condition effect, Knobel, Trans. Am. Electrochem. Soc. **47** (1925), Alternating Current effect, Goodwin and Knobel, *ibid.*, **37** 553 (1920).

[3] Only approximately constant, but no great precision is claimed for the dependent calculations.

and at equilibrium respectively, we can insert Equation (1) and obtain

$$E = \frac{mRT}{nF} ln \frac{H_a}{H_a'} \tag{3}$$

where H_a and H_a' represent the excess and equilibrium surface concentrations respectively. It is seen that the coefficient of the logarithm term is now greater than $\frac{RT}{nF}$ and there are many cases in the literature where it has been found necessary to empirically make this coefficient larger in order to fit experimental data.

The validity of these assumptions will not be discussed here, but will be judged by their ability to explain the overvoltage data.

The concentration measurements quoted yielded values of the order of 10^{-11} mols of H_1 per square centimeter of electrode surface, for low polarization running up to 10^{-9} to 10^{-8} mols at higher current densities, for smooth electrodes. These figures correspond to about 1/100 to 1 molecular layer. (On platinized platinum where the adsorbing surface is spread over an appreciable depth, much larger concentrations were found.)

As to the state of the adsorbed gas, and the question of compound formation, the author subscribes to the excellent discussion of Evans[4] in his article on passivity. Evans makes the point that these so-called compounds cannot be considered compounds in the usual sense in that the percentage composition varies continuously with the hydrogen or oxygen pressure and with the depth below the surface. Absorbed hydrogen on platinum may be called a semi-compound, lacking the definite composition of a true compound, and yet the two elements are united to a certain extent by presumably the same kind of forces that result in compounds.

The detailed mechanism of the action of the hydrogen electrode is believed to be as follows: On bubbling H_2 about a piece of platinized platinum, the hydrogen is adsorbed and the simultaneous dissociation which takes place is a concurrent feature of the adsorption. This process continues until equilibrium is attained, the pressure of H_1 being about 10^{-23} *At* (as calculated from the free energy of dissociation) but the concentration of H_1

[4] Trans. Far. Soc. **18** 6 (1922).

being relatively large due to its being held to the electrode. That is, due to the adsorption the surface concentration is greater than if there were no adsorption. If now the electrode be made slightly negative, H^+ ions discharge from the solution, and in so doing pass through the adsorbed state, that is, while taking on their electrons. This tends to produce an excess concentration of H_1, which gives rise to the measured overvoltage. The excess concentration is always being reduced by the pairing of atoms to form molecules which no longer affect the electrode potential and eventually leave in bubbles. As the current increases, the excess concentration tends to become greater and will become sufficiently large so that the rate of loss of H_1 to form H_2 gives a dynamic balance with the current. On making the electrode slightly positive with respect to the solution, the adsorbed hydrogen atoms tend to ionize, but the surrounding H_2 is rather slow in dissociating to form more H_1, so high polarization arises for small currents. Exactly the same phenomena occur on a smooth lead electrode, for example, only here the concentrations of adsorbed gas are much smaller. That a smooth lead electrode will *tend* to act as a platinized platinum hydrogen electrode is shown by the fact that it approaches zero potential after the polarizing current is shut off. (See curves of previous article.) That it does not act as a satisfactory hydrogen electrode is due merely to the small quantity of H_1 available on the surface to give the desired current. As this lead electrode is made more negative, the current increasing, the H_1 concentration starting from about 1/100 of a molecular layer, increases, tending toward a molecular layer. On account of the small adsorptive power of lead for hydrogen a small increase in concentration is accompanied by a comparatively large increase of pressure and consequently of overvoltage. Or according to Freundlich's Equation (1) the exponent m is large and makes the overvoltage from Equation (3) large. That the concentration should not exceed a molecular layer is in accord with modern ideas of adsorption. The formation of a second layer is probably not possible, for it is only on account of the mutual force between the hydrogen atoms and the metal that the atoms do not pair off as fast as in free space; and the range of action of the metal probably does not reach as far as to the second layer. It is possible that

the ions are discharged while in the second layer but in that case they would unite almost instantly with some other atom.

To speak in terms of energy, the loss represented by the product of the overvoltage and the number of coulombs, arises from the fact that H_1 is going to form H_2 from a pressure greater than its equilibrium value. The only limit is when the H_1 pressure has reached 1 atmosphere, or that of the whole system. This high pressure may be approached when the ions are being discharged in the second layer, for even though their life as atoms be extremely short, the energy loss results. The free energy of formation of H_2 at one atmosphere from H_1, at one atmosphere is not accurately known but is of the order of 65,000 calories[5] at 25°. This corresponds to an overvoltage of about 1.4 volts and constitutes the maximum possible hydrogen overvoltage under the above conditions.[6]

Observation of the overvoltage current density curves[7] for hydrogen show that a large majority do tend toward 1.2 to 1.3 volts, but that none exceed the value 1.4 volts. By the same reasoning, we would expect the occurrence of higher oxygen overvoltages and that the maximum halogen overvoltages would be smaller. This is qualitatively true, but the data do not point to any such definite maxima as for hydrogen overvoltages.

Effect of different electrode metals

The well-known and decided effect of the electrode metal, on the value of the hydrogen overvoltage, shows up in Equation (3) in the factor m. Comparing two metals it is probable the better adsorber will increase both H_a and H_a' of Equation (3) by approximately the same percentage amount and leave the ratio constant; constant, that is, compared to the enormous variation it would have to have if the whole variation on various electrodes were to be explained by it. The factor m, the exponent of Freund-

[5] Estimated from the value of Langmuir, Jour. Am. Chem. Soc. **37** 417 (1915) and of Bohr, Abhandlungen uber Atombau, p. 58 Vieweg 1921. Lewis and Randall "Thermo-dynamics" give a higher value.

[6] Bennett and Thompson, Trans. Am. Electrochem. Soc. **29** 273 (1916) calculated a similar maximum overvoltage but used a different value for the heat of reaction and neglected to calculate the free energy. Rideal, Jour. Am. Chem. Soc. **42** 94 (1920) has also arrived at a similar figure for the maximum overvoltage but by an entirely different method.

[7] Knobel, Caplan and Eisemann, *loc. cit.*

lich's Equation (1), is small for metals which are good adsorbers of the gas and large for poor adsorbers. This results in high overvoltage on poor adsorbing metals, and vice versa, which fact has been long known to be true in a qualitative way. In Table I a few values of m, derived from the time overvoltage curves are listed beside the corresponding overvoltages, each taken at the arbitrary current density of 0.01 ampere per square centimeter.

TABLE I

Electrode	*H_2 overvoltage* 0.01 *amp./sq.cm.*	*m(approx.)*	*ov./m.*
Platinized Pt	0.032	1	.032
Smooth Ni	0.469	7.1	.067
Smooth Cu	0.506	6.6	.077
Smooth Ag	0.584	7.7	.075
Smooth Pb	0.890	11.	.088
Smooth Hg	1.066	15.	.071

In the last column the ratios of the overvoltages to m are tabulated. Since m is only known approximately the constancy of the ratios is entirely satisfactory to support the statement that the overvoltage depends mainly on m.

Similarly the difference between a rough and smooth surface of the same metal is explained by the greater adsorbing power and smaller value of m for the rough surface, giving rise to a lower overvoltage.[8]

Effect of current density

From Equation (3) we can, with certain simple assumptions, derive an equation for overvoltage as a function of the current density. Consider an electrode with a constant current density I amperes per square centimeters, depositing hydrogen at an overvoltage E. The rate of increase of concentration of monatomic hydrogen on the electrode surface due to the current will be proportional to the current and given by

$$\frac{dH_a}{dt} = k_1 I \tag{4}$$

[8] The data and further explanation of this feature are given in the paper on the Surface Condition Effect previously quoted.

where k_1 is mols of H_a per coulomb and equals 1/96,500. The monatomic hydrogen is lost by the reaction

$$2H_a \rightarrow H_2 \tag{5}$$

and with the assumption that this reaction follows the usual reaction velocity equation we have

$$\frac{-dH_a}{dt} = k_2(H_a)^2 - k_3(H_2) \tag{6}$$

where k_2 and k_3 are reaction velocity constants. The last term is due to dissociation of H_2 and under most conditions will be a negligible factor. At the steady state of electrolysis, the net change of concentration of H_a must be zero, whence

$$k_1 I = k_2(H_a)^2 - k_3 H_2 \tag{7}$$

and

$$(H_a)^2 = \frac{k_1}{k_2} I + \frac{k_3}{k_2} H_2. \tag{8}$$

Grouping constants, and remembering that the H_2 pressure is held constant at one atmosphere

$$H_a = \text{const.} \sqrt{I + k_4}. \tag{9}$$

Equation (9) may now be inserted in Equation (3), whence

$$E = \frac{mRT}{nF} ln \frac{\text{const.} \sqrt{I + k_4}}{H_a'}. \tag{10}$$

Since H_a', the equilibrium concentration for a given H_2 pressure, is constant, another regrouping of constants may be made and

$$E = \frac{mRT}{2nF} ln(I + k_4) + k_5. \tag{11}$$

This equation is of the correct form to represent the experimental data provided the current density be not too large. At the higher current densities, where the maximum overvoltage and a saturation of the surface by a complete molecular layer is approached, it is to be expected that the exponent m of Freundlich's Equation (1) will change. The precision with which m may be evaluated from Equation (11) is very poor on account of the way in which the unknown constant k_4 enters and which must be determined empirically. However, the value of m is approximately the same

as that derived by other methods, and it is important to note that without its introduction into the coefficient of the logarithm of Equation (11), that equation would be totally incapable of fitting the experimental data. On account of the necessity of empirically determining the constants of Equation (11) and of their uncertainty when determined, due partly to the irreproducibility of the measurements, only a few values of m will be given. For hydrogen overvoltage on platinized platinum, copper and lead, the quantity m is 1, 6.7, and 13 respectively as determined from the commutator measurements previously mentioned, and 0.8, 4 and 11 from current density data.

It is unfortunate that more reproducible overvoltage measurements are not possible in order that more exact quantitative tests may be made of the theory. The irreproducibility is probably not due to poor experimentation, however, but to the inherent instability of the phenomenon involving as it does only the few molecules in a partial molecular layer on the surface. One small bubble originally on the surface might on leaving considerably disturb the dynamic equilibrium around that point and make a large change in the overvoltage. Also a slight change in the character of the electrode surface would likely affect the overvoltage considerably. It is probable that we must be satisfied with the best average values of overvoltage that can be obtained from many experiments, and in making tests take into account the variability.

Effect of pressure

The change of gas overvoltages with pressure may be predicted and experiments confirm the result. In the general equation for overvoltage

$$E = \frac{mRT}{nF} \ln \frac{H_a}{H_a'} \tag{12}$$

the only term which varies appreciably with the pressure is H_a', the concentration of adsorbed gas at equilibrium pressure. H_a is essentially constant for a constant current since its value is determined by a balance between the rate of production by the current and loss by pairing of atoms and these effects should be

independent of the pressure. The reverse reaction of dissociation of H_2 does depend on the H_2 pressure but occurs to such a slight extent that it cannot change appreciably the atom concentration, H_a.

The equilibrium pressure P' of atomic hydrogen is proportional to the square root of the diatomic hydrogen pressure P_{H_2}, from the equilibrium constant, whence assuming Freundlich's equation holds again for the relation between P' and H_a' we have

$$P_{H_2}^{\frac{1}{2}} = \text{const } (H_a')^m. \tag{13}$$

Substituting (13) in (12)

$$E = \frac{mRT}{nF} ln \frac{H_a}{\text{const } (P_{H_2})^{\frac{1}{2}m}} \tag{14}$$

and representing the constant part by a symbol E_0

$$E = E_0 - \frac{RT}{2F} lnP_{H_2}. \tag{15}$$

This equation shows that the overvoltage should increase with decrease of the pressure of H_2 and by a calculable amount from its value E_0 at atmospheric pressure. The reversible hydrogen electrode also changes with pressure and by the same amount $\frac{RT}{2F} lnP_{H_2}$, but decreases in potential with decrease in pressure. If, therefore, the electrode potential at an electrode showing overvoltage, is measured with a reference electrode as the mercury, mercurous sulphate, whose e.m.f. does not depend appreciably on the pressure, the measured potential should remain constant and equal to E_0 as the pressure is changed. This result is confirmed by measurements of Bircher and Harkins[9] and Knobel[10] on many electrodes and at various constant current densities for pressures from atmospheric down to about 2 cms. of mercury. One measurement of Newbery's[11] at pressures above atmospheric is available. He measured the oxygen overvoltage on platinum at 100 atmospheres, and finds a decrease of overvoltage of about 0.02 volt (measured directly against a standard oxygen electrode). The

[9] J. A. C. S. **45** 2890 (1923).
[10] J. A. C. S. **46** 2751 (1924).
[11] Jour. Chem. Soc. **105** 2428 (1914).

calculated decrease is $\frac{RT}{4F} ln\ 100 = 0.015$ volt. This is as close a check as could be expected with the use of a reference electrode as uncertain as the oxygen electrode.

Relation to surface tension

The relation of overvoltage to the surface tension of the solution can be accounted for entirely by the pressure changes which occur in small bubbles due to surface tension. The pressure inside a bubble of radius r, surrounded by a liquid of surface tension γ is greater than the pressure on the liquid by $\frac{2\gamma}{r}$ and this term may be of appreciable magnitude if the bubble radius be very small. Attempts have been made to account for all gas overvoltages by this effect.[12] It is inadequate, however, for even in the extreme case of a bubble no larger than a molecule, the resulting overvoltage would be only about 0.12 volt. There is some question whether this effect should properly be called overvoltage, for the change in potential due to increased pressure in the small bubble is no different from the change which would result from applying an external pressure of the same magnitude.

Other factors determining overvoltage

The experimentally observed decrease in overvoltage with increase in temperature is easily explained qualitatively by the increase in the reaction velocity of the atoms to form molecules, thus decreasing the concentration of atoms adsorbed on the electrode. A quantitative prediction does not seem possible for lack of information as to the temperature coefficient of the reaction velocity.

The effect of time, in the case of some electrodes which reach a maximum constant overvoltage only after several hours; the effect of stirring the solution in lowering the overvoltage, and the effect of colloids in raising the overvoltage, all receive a qualitative explanation on the basis of an adsorbed atomic gas layer as the primary cause of overvoltage. Details will not be given since no quantitative tests can be made as yet.

[12] See MacInnes and Adler, Jour. Am. Chem. Soc. **41** 194 (1919) also Thiel and Brenning, Zeit. anorg. Chem. **83** 229 (1913).

Summary

The postulates of this theory of overvoltage are (1) that atomic gas is the electromotively active substance and that the molecule has only an indirect effect by its influence on the atomic gas pressures; (2) that overvoltage is due to an excess surface concentration of this adsorbed atomic gas, and that this surface concentration is connected with the pressure by an equation of the form of Freundlich's isotherm.

From these two postulates a mechanism of the action at polarized electrodes is obtained, which involves reasonable concentrations of the active substances at all overvoltages. A maximum overvoltage is calculable which agrees with the experimental values for hydrogen overvoltages, and is not in disagreement but cannot be quantitatively tested for other gases. The effect of different electrode materials and their surface condition on overvoltage is shown to be related to the adsorbing power of the material. An equation for the effect of current density is deduced which approximately represents the experimental data. That the check is not better is believed to be due to the inherent instability of the surface film, precluding accurate reproducible measurements. The equation differs from others in that the coefficient of the logarithmic term is much larger than the usual $\frac{RT}{nF}$ as it must be to even approximate the facts. The effect of pressure on overvoltage is accounted for and agrees with experiment. The theory also explains qualitatively the effect on overvoltage of other factors such as stirring, temperature and colloids, although details are not given for lack of data for quantitative test.

(Contribution from the Rogers Laboratory of Physics, Electrochemical Laboratory, Massachusetts Institute of Technology.)

THE PENETRATION OF ELECTROLYTIC HYDROGEN THROUGH IRON

By Max Knobel and Ralph B. Norton

In the preceding paper by one of the authors, a theory of overvoltage is described, and tested by the result of experimental work on gas overvoltages. In the present paper the results of experiments on the penetration of electrolytic hydrogen through iron are presented, which give further evidence in support of this theory.

The experimental facts in regard to hydrogen penetration are as follows: (1) Iron[1] and palladium[2] in particular, and other metals at sufficiently high temperatures are permeable to hydrogen gas. Palladium is appreciably permeable at room temperature and iron begins to be permeable at about 325° C. These metals are not appreciably permeable, however, to nitrogen or oxygen. (2) Making an iron or palladium tube a cathode[3] in a solution at room temperature causes the discharged hydrogen to penetrate to the inside of the tube where comparatively high pressures (26 atmospheres) may be built up without any noticeable slowing down of the penetration speed. (3) A tube of iron merely immersed in an acid or alkaline solution will accumulate hydrogen on the inside. (4) Increasing the temperature in any of the above cases will cause a marked increase in the speed of penetration. (5) The physical properties of the iron or steel when subjected to hydrogen as above are injuriously affected.[4] The iron becomes more brittle and while its tensile strength may not be decreased the weakening shows up in bending tests. (6) In electroplating or pickling iron where it must necessarily be subjected to hydro-

[1] See Charpy and Bonnerot, Compt. Rend. **154** 592 (1913), **156** 394 (1914).
[2] See Winkelmann, Ann. Physik. **6** 104 (1901).
[3] Fuller, Trans. Am. Electrochem. Soc. **36** 113 (1919). Bodenstein, Zeit. Elektrochem. **28** 517 (1922).
[4] Heyn, Stahl und Eisen **20** 837 (1900), Zeit. phys. Chem. **58** 760 (1907). Stromeyer, J. Iron and Steel Inst. 1 (1909). Wedding, Int. Cong. App. Chem. **2** 27 (1903). Andrew, Trans. Far. Soc. **9** 316 (1914). Baraduc-Muller, Rev. Met. **12** 804 (1915). Merica, Met. Chem. Eng. **16** 496 (1917).

gen, brittleness is produced.[5] (7) The brittleness can be more or less completely relieved by heating even as low as 100° C. and more quickly at higher temperatures.[6] In the process of heating considerable hydrogen is given off.

The reason for the penetration and the essential feature of its mechanism is almost universally accepted to be as follows:[7] By thermal dissociation of H_2 or by cathodic discharge or chemical reaction, a concentration of monatomic hydrogen is produced on one side of the metal sheet which atomic gas can pass through the metal and on emergence into the gas phase on the other side reforms diatomic hydrogen. There is good evidence that monatomic hydrogen is present in those cases where penetration occurs. In electrolytic hydrogen, H_1 is without doubt present, and the hydrogen produced when acid or alkali acts on iron is generally believed to be " nascent " or monatomic. When free gas only is on one side of the penetrated sheet, the temperature must be increased (tending to produce dissociation) in order to attain appreciable penetration. Perhaps a more convincing proof that it is the atomic gas which is penetrating is that chromate or dichromate in the solution about an iron tube will prevent the penetration.[8] Diatomic hydrogen would not be oxidized by dichromate but atomic hydrogen very likely would be.

Accepting the above explanation of the hydrogen diffusion, it is apparent that the rate and magnitude of the penetration should be very intimately connected with the hydrogen overvoltage on the outside surface. Bodenstein,[9] in this connection, has attempted

[5] Fuller, Trans. Am. Electrochem. Soc. **32** 247 (1917). Coulson, Trans. Am. Electrochem. Soc. **32** 237 (1917). Thompson and Richardson, Met. Chem. Eng. **16** 83 (1917).

[6] Heyn, Stahl u. Eisen (1908).

[7] The following investigators endorse this or essentially this explanation: Bodenstein, Fuller, Merica, Coulson, Charpy and Bonnerot, Andrew, Richards and Behr, Winkelmann, Sieverts, Zeit. phys. Chem. **77** 591 (1911), Parr. Univ. Ill. Eng. Expt. Sta. Bull. No. 94, (1917), Guillet J. Iron and Steel Inst. **90** 66, (1914) Hoitsema, Zeit. phys. Chem. **17** 1 (1895) and others. Some confine their work to adsorption only rather than penetration and in that case consider the adsorption to be of the atomic gas. Thompson and Richardson (*loc. cit*) say the embrittlement of their springs is not due to hydrogen evolution, but hydrogen was certainly present during the plating even if not visibly evolved. MacInnes Trans. Am. Electrochem. Soc. **36** 136 (1919) believes that diatomic hydrogen is penetrating but it is difficult to see how high internal pressures can be developed if that be true.

[8] Fuller, Trans. Am. Electrochem. Soc. **36** 126 (1919) Merica, *loc. cit.*

[9] Bodenstein, Z. Elek. **28** 517 (1922).

to determine how the rate of penetration of hydrogen through iron depends on the hydrogen overvoltage. He deduced an equation between overvoltage and penetration rate briefly as follows: The overvoltage E may be expressed by the usual equation[10]

$$E = \frac{RT}{F} \ln \frac{P_H}{P'_H} \tag{1}$$

where P_H and P'_H are the partial pressures of monatomic hydrogen under overvoltage conditions and at equilibrium respectively. Since P'_H depends only on the pressure of H_2 if the latter be held constant at one atmosphere, Equation (1) can be written

$$E = 0.059 \log P_H + \text{const.} \tag{2}$$

where numerical values for 25° are substituted. Bodenstein now assumes that the concentration of atomic hydrogen C_H is proportional to P_H and that the penetration rate D is proportional to C_H, whence

$$E = 0.059 \log D + \text{const.} \tag{3}$$

D may be expressed in c.c. per hour or any other such unit and the additive constant contains the other constant factors on which the diffusion is dependent, as area, thickness of the sheet, etc.

The author's previous work[11] has shown that it is incorrect to assume pressure and concentration of this adsorbed gas to be proportional, but rather that an equation of the form of Freundlich's must be used, namely

$$P_H = \text{const.}\ C_H^m \tag{4}$$

where m is approximately constant for any one metal for not too great a concentration range. Using Equation (4), therefore, to introduce penetration rate in Equation (2) and otherwise using Bodenstein's reasoning, we obtain

$$E = 0.059\ m \log D + \text{const.} \tag{5}$$

Bodenstein's experimental results are in accord with Equation (5) but not with his own Equation (3). He finds that instead of the value 0.059, he must use values of this coefficient ranging from

[10] Bodenstein's method is not reproduced exactly, but no essential change is made. The reason for not adhering exactly to his procedure is to bring out more clearly the relation of his equation to another which will be given.

[11] Knobel, Jour. Am. Chem. Soc. **46** 2613 (1924) and preceding article.

0.1 to 0.9. The reproducibility of his penetration rates is very poor as shown by the variation of this coefficient so that no precise evaluation of m can be made from his data. To justify Equation (5) it must be shown that m is approximately constant and furthermore that the value obtained from these penetration experiments is equal to that obtained in two other ways, which will be explained later. For these reasons, the experimental work has been repeated with some modifications in apparatus and procedure.

Experimental

The apparatus is shown in Fig. 1. The steel tube A is of $\frac{5}{16}$-inch diameter, 22-gauge Shelby seamless tubing turned down to leave

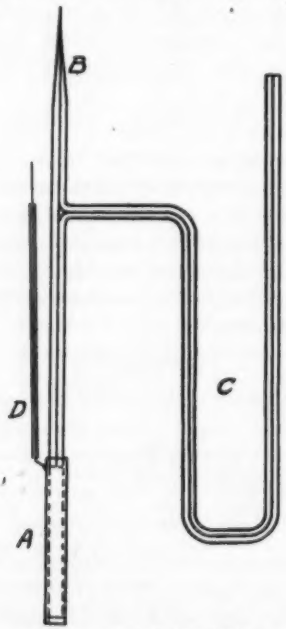


Fig. 1. Penetration Unit

a wall thickness of about 0.016 inch. The full 22 gauge wall was found to give too slow penetration and rather erratic results.

The length is 3 inches, and the bottom sealed by brazing in a small iron plug, the brass then being covered with asphalt. Electrical connection to the steel is made by soldering on a copper wire *D* which was later insulated by an asphalt coating. To the top of the steel tube is connected the 3 mm. glass capillary of the form shown, the connection being made tight with asphalt. The end *B* is drawn out in a very fine long capillary and is used to let out accumulated hydrogen on occasion. The tip was broken and resealed each time; the capillary being so small that no appreciable volume change resulted. The U-tube *C* contains mercury and serves as a monometer to measure the pressure of gas which penetrates through the tube. These units are placed in the center of a beaker containing 2N, H_2SO_4 and a cylindrical lead anode, and the whole apparatus immersed, as far as possible, in a thermostat to keep the temperature constant at 25° C. The hydrogen overvoltage was measured at the tube surface by a Hg, Hg_2SO_4 reference electrode in the usual way.

The reproducibility of the cathodic penetration of hydrogen through iron is very poor as found by Bodenstein and others, so that several similar pieces of apparatus must be run together. For the present purpose of testing Equation (5), however, it is not necessary that each apparatus give the same absolute penetration rate but only that the rate be constant at a given overvoltage and vary with change in the overvoltage in the same way for each, so that m (of Equation (5)) is constant. Six units were run together, all at the same current density (same overvoltage) and then together changed, to other values of current density or overvoltage. Readings were taken every few minutes of overvoltage and pressure developed in the tube. This pressure may be taken as a measure of the penetration for the volume is nearly constant.

Some of the experimental data are given in Table I for three of the units. The other three units developed leaks before all the measurements were completed and although the first values from them approximately check those from the other three, they are discarded from the computation. The first column gives the constant current density of electrolysis. During the course of a run, which lasts from one-half to several hours, depending on the

current density, the overvoltage may change by a few hundredths of a volt. From readings taken every few minutes, an average value is computed and is given in the second column. The last column contains the penetration rate as pressure developed in millimeters of mercury per minute. This figure is also an average for the whole run but, except for the first few minutes, the penetration rate is fairly constant. Both this average and that of the overvoltage were determined graphically. The absolute penetration rate was not found in most cases but is of the order of 4 to 10 c.c. per day.

Using the data of Table I, a plot is made of the logarithm of the penetration rates against the overvoltage. Through these points, for each unit, the best straight line is drawn and its slope graphically determined. The slope should equal 0.059 m (see Equation 5) from which m is computed. In these plots the points do not lie closely to a straight line but the deviations are such as to indicate no systematic deviation from a straight line but rather experimental errors. The slope may not, therefore, be determined to better than about 10 per cent. The values found are 0.35, 0.34 and 0.33 for the slope whence $m = 5.8$ (average).

This quantity m, is the same as that appearing in the overvoltage time equation previously published[12] and in the current density overvoltage equation of the preceding article. The overvoltage time measurements were made on both electrolytic iron and on a piece of wrought iron, and while these samples may differ somewhat from the steel tubing used here, the results should be nearly comparable. The average value of m from three overvoltage time curves (including two for electrolytic and one for wrought iron) is 5,1.

[12] Knobel, Jour. Am. Chem. Soc. **46** 2613 (1914.)

TABLE I

Penetration of Hydrogen through Iron

Current in amps.	*Overvoltage in volts*	*Penetration mm. Hg per min.*
	Apparatus No. 1	
1.0	0.980	0.516
0.1	0.765	0.10
1.0	0.945	0.467
0.05	0.715	0.109
3.0	0.930	0.838
0.01	0.590	0.050
0.01	0.620	0.0392
	Apparatus No. 2	
1.0	0.918	0.440
0.1	0.725	0.0715
1.0	0.863	0.440
0.05	0.727	0.179
3.0	0.883	0.540
0.01	0.611	0.095
0.01	0.645	0.086
	Apparatus No. 3	
1.0	0.907	0.680
0.1	0.776	0.161
1.0	0.843	0.440
0.05	0.787	0.241
3.0	0.843	0.880
0.01	0.665	0.215
0.01	0.741	0.157

To determine m from the current density equation, determinations were made of the overvoltage at various current densities on one of the penetration units. (The data which might be taken from the penetration experiments cannot be used as these lasted over several days and the overvoltage at one current one day can hardly be compared with that for another current at a later day on account of changes in the surface.) Table II gives the mean

values of several runs. To obtain m, these data should be empirically fitted to the equation

$$E=\frac{mRT}{2F}ln\ (I+\text{const}_1)+\text{const}_2 \qquad (6)$$

given in the preceding article. As mentioned there, the precision attainable is very poor, but by neglecting const_1 and plotting $1nI$ against E an approximate value may be obtained. This method yields $m=6.4$ using only the lower current densities where the equation is probably more accurate and gives $m=9.0$ from the line drawn to include all current densities.

TABLE II

Overvoltage Current Density Values

Current Density *amps. per sq. cm.*	*Overvoltage* *volts*
0.000633	0.460
0.00633	0.608
0.0190	0.693
0.0633	0.907
0.1265	0.976
0.190	1.017

The approximate concordance of these three values of m, 5.8, 5.1, and 6.4 determined by three different methods is believed to be important support for the theory of overvoltage on which the formulae are deduced. The values from the current density and the time-overvoltage equations may be somewhat dependent in that the deduction of the equations and the measurements are somewhat similar, but the penetration experiments are certainly of a different nature and provide an independent check on the mechanism postulated.

It is worth noting that without the introduction of Freundlich's equation and the exponent m, that is, assuming as Bodenstein and others have, that the pressure and concentration of gas are proportional, one would expect the penetration rate to change by many powers of ten for a small change in overvoltage. From the adsorbed gas concentrations previously mentioned and the present theory of overvoltage, however, the rate should not change by

more than 100 to 1 at the most, corresponding to whether the surface concentration be 1/100 or up to 1 complete molecular layer, for zero and maximum overvoltage respectively.

The results of this work are therefore two in number, (1) the mechanism of the penetration of cathodic hydrogen through iron, and its dependence on the hydrogen overvoltage, as previously given, is confirmed. (2) A further justification is provided for the introduction of the exponential relation between concentration and pressure of adsorbed hydrogen on polarized electrodes, and the overvoltage theory based on this postulate is supported.

Summary

An equation for the rate of penetration of cathodic hydrogen through iron as a function of the hydrogen overvoltage is deduced from a theory of overvoltage. Experimental values for the penetration rate are given and found to be in accord with the predicted values as far as variation with overvoltage is concerned. The generally accepted mechanism of the penetration is confirmed and the results support the theory of overvoltage on which their explanation is based.

A NEW FORMULATION OF THE LAWS OF QUANTIZATION OF PERIODIC AND APERIODIC PHENOMENA

By MAX BORN[1] AND NORBERT WIENER

The representation of the quantum laws by matrices[2] incurs serious difficulties in the case of aperiodic phenomena. For example, in the extreme case of uniform rectilinear motion, since no periods are present, the coördinate matrix q can have no element outside the diagonal $m=n$. However, even if m and n are continuous, this is not possible in any proper sense. We shall consequently try to generalize the algorithm of the quantum rules in such a manner as also to embrace these singular cases. We shall show in what follows how this can be done, and shall illustrate our method of attack by a few simple examples.

§1. **The calculus of operators.** It has been found advantageous in the theory of perturbations to consider the matrix q as an "operation"; that is, as a linear transformation of an array of variables $x_1, x_2, \ldots$, such as

$$y_m = \sum_n q_{mn} x_n \tag{1}$$

This is more especially the case in the treatment of singular or degenerate instances. The multiplication of matrices has a direct interpretation as the successive application of the two corresponding transformations.

In place of the transformation of an array of variables $x_1, x_2, \ldots$, which lack any very obvious physical interpretation, we may consider transformations of functions $x(t)$ of the time. Let us assume that x possesses the series development

$$x(t) = \sum_n x_n e^{\frac{2\pi i}{h} W_n t}. \tag{2}$$

[1] Professor of Theoretical Physics at the University of Göttingen, Foreign Lecturer at the Massachusetts Institute of Technology for the Fall Term, 1925–26.

[2] Cf. W. Heisenberg, ZS. f. Phys. Vol. 33, p. 879, 1925. M. Born and P. Jordan, *ibid.* M. Born, W. Heisenberg, and P. Jordan, *ibid.*

We may then invert this transformation of the variables x_n, and obtain these from $x(t)$ by the process of forming means:

$$x_n = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} x(s) e^{-\frac{2\pi i}{h} W_n s} ds. \tag{3}$$

We may form in a like manner from the quantities y_n the function

$$y(t) = \sum_m y_m e^{\frac{2\pi i}{h} W_m t}.$$

If we now perform the substitution indicated in (1) and then employ (3), we get

$$y(t) = \sum_{mn} q_{mn} \lim_{T \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} x(s) e^{-\frac{2\pi i}{h} W_n s} ds \, e^{\frac{2\pi i}{h} W_m t},$$

or if we rearrange the terms,

$$y(t) = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} q(t,s) x(s) ds \tag{4}$$

where

$$q(t,s) = \sum_{mn} q_{mn} e^{\frac{2\pi i}{h}(W_m t - W_n s)}. \tag{5}$$

That is, the " operator "

$$q = \left(\lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} ds \; q(t,s) \; \ldots \right) \tag{6}$$

transforms the function $x(t)$ into the function $y(t)$. We denote this by

$$y(t) = qx(t) \tag{7}$$

It is, however, true that the representation of $q(t, s)$ by the series (5) will generally diverge, as for example in the case of the harmonic oscillator, where the terms $q_{m,m\pm1}$ become infinite as $\sqrt{m}$.

We shall consequently entirely disregard in the sequel the explicit representation of $q(t, s)$ by a trigonometric series. We shall therefore admit all functions $q(t, s)$ as possible generators of the operator q. Indeed, we shall go further, and not consider an operator as necessarily defined by an integral mean such as (6), for even this formula may fail us. In its place, we make the following general definition: *An operator q is a rule in accordance with which we may obtain from a function $x(t)$ another function $y(t)$, which we symbolize by $y(t) = qx(t)$. The operator q is linear if*

$$q(x(t) + y(t)) = qx(t) + qy(t). \tag{8}$$

The operation of multiplying a function $x(t)$ by a function $f(t)$ is accordingly a linear operator, which we shall symbolize by $(f(t))$, or in cases where there is no danger of ambiguity, by $f(t)$ alone. This operator when applied to $x(t)$ yields $y(t)=f(t)x(t)$.

The successive performance of two operations is itself an operator, which we shall write as the operational product of the two operators. The operator pq, that is, represents the result of first performing q on a given function, and then p. Thus $pqx(t)=p(qx(t))$. Of course, this multiplication is in general not commutative.

The unit or identity operator 1 leaves every function unaltered.

There corresponds to every operator a "matrix." In order to understand the meaning of the necessary generalized definition of a matrix, we again depart from the representation of an operator in terms of a generating function in accordance with formula (6). If we apply this operator to the function $x(t)=e^{\frac{2\pi i}{h}W_k t}$, we obtain in accordance with (4)

$$y_k(t)=\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}\sum_{mn}q_{mn}e^{\frac{2\pi i}{h}(W_m t-W_n s+W_k s)}ds=\sum_m q_{mn}e^{\frac{2\pi i}{h}W_m t}.$$

Consequently, since $h\nu(nm)=W_k-W_m$, we have

$$q_k(t)=e^{-\frac{2\pi i}{h}W_k t}y_k(t)=\sum_m q_{mk}e^{\frac{2\pi i}{h}(W_m-W_k)t}=\sum_m q_{mk}e^{2\pi i\,\nu(mk)t} \tag{9}$$

as the sum of the elements of a line of the coördinate matrix. This function will be complex even when the matrix (q_{mn}) is Hermitian. We may, nevertheless, conceive $q_k(t)$ as in a certain sense the motion belonging to the kth quantum state. There are then two real motions belonging to every state, corresponding respectively to the real and pure imaginary parts of the line of the matrix. If $q_k(t)$ is real, it defines the motion in the ordinary sense.

By the harmonic analysis of $q_k(t)$, we obtain the elements q_{mn} of the matrix (q_{mn}):

$$q_{nk}=\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}q_k(t)\,e^{-2\pi i\nu(nk)t}dt. \tag{10}$$

We generalize these rules as follows: *The operation* $e^{-\frac{2\pi i}{h}Wt}q$, *applied to the function* $e^{\frac{2\pi i}{h}Wt}$, *generates the function*

$$q(t, w) = e^{-\frac{2\pi i}{h}Wt} q e^{-\frac{2\pi i}{h}Wt} \tag{11}$$

We call this the sum of the line of the " matrix " of q corresponding to the value W of the energy. By the harmonic analysis of $q(t, W)$, *we obtain the elements of the " matrix "*

$$q(V, W) = \lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau} q(t, W)e^{-\frac{2\pi i}{h}(V-W)t}dt = \lim_{\tau\leftarrow\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau} e^{-\frac{2\pi i}{h}Vt} q e^{\frac{2\pi i}{h}Wt}dt. \tag{12}$$

The function $q(t, W)$ may exist in cases where the " matrix " $q(V, W)$ does not exist in the proper sense. In these cases, $q(t, W)$ does not admit of a harmonic analysis in the proper sense. We may, nevertheless, employ the methods of generalized harmonic analysis to isolate the elementary harmonic components. We may say in the same sense as above that $q(t, W)$ represents the "motion with energy W."

Among the linear operators are differentiation and integration:

$$D = d/dt, \quad I = \int dt.$$

We shall be especially concerned here with the operator D. By combination with an arbitrary operator q, we can generate the products Dq and qD. In order to discover the meaning of these operators in the language of matrices, we suppose q to be the operator defined by (5) and (6). It is then obvious that Dq transforms $x(t)$ into

$$y(t) = Dqx(t) = \frac{d}{dt}\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau} q(t, s)x(s)ds$$

$$= \lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}\frac{\partial q(t,s)}{\partial t}x(s)ds.$$

That is, Dq is also an operator of the special form (6) with the generating function

$$\frac{\partial q(t, s)}{\partial t} = \frac{2\pi i}{h}\sum_{mn} q_{mn}W_m e^{\frac{2\pi i}{h}(W_m t - W_n s)}$$

The corresponding matrix is $\left(\frac{2\pi i}{h}W_m q_{mn}\right) = \frac{2\pi i}{h} Wq$, where W denotes the diagonal matrix $(W_n \delta_{mn})$.

qD transforms $x(t)$ into

$$z(t) = qDx(t) = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} q(t,s)x'(s)ds,$$

where $x'(t)$ denotes the derivative of $x(t)$. If we assume that there is no contribution from the limits as they recede infinitely, we obtain on integration by parts the expression

$$z(t) = qDx(t) = -\lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} \frac{\partial q(t,s)}{\partial s} x(s)ds.$$

Thus qD is also an operator of the form (6) with the generating function

$$\frac{-\partial q(t,s)}{\partial s} = \frac{2\pi i}{h} \sum_{mn} q_{mn} W_n e^{\frac{2\pi i}{h}(W_m t - W_n s)}$$

and with the corresponding matrix $\left(\frac{2\pi i}{h} W_n q_{mn}\right) = \frac{2\pi i}{h} qW$.

It follows from this that the operator D belongs to the matrix $\frac{2\pi i}{h}W$, and furthermore, that the operator $Dq - qD$ corresponds to the matrix $\dot{q} = \frac{2\pi i}{h}(Wq - qW)$ with the elements

$$\frac{2\pi i}{h}(W_m - W_n)q_{mn} = 2\pi i \nu(m,n) q_{mn}.$$

We consequently define the time derivative $\dot{q}$ of an operator q as the operator

$$\dot{q} = Dq - qD. \tag{13}$$

As in the case of any other operator, we may form from D the powers $D^2, D^3, \ldots$, and consequently polynomials $f(D)$. We may proceed from these by limit processes to more general functions $f(D)$, such as $\sqrt{D}$, or D^n with an arbitrary exponent. The more rigorous justification of this theory is to be found in a number of special memoirs.[3] Here we shall only make use of the fact that

[3] Cf. S. Pincherle, Funktionaloperationen und -Gleichungen, Ency. d. math. Wissenschaften, II A 11; N. Wiener, The Operational Calculus, forthcoming in Math. Annalen.

the result of applying the operator D^n to the function $e^{i\omega t}$ gives $(i\omega)^n e^{i\omega t}$ for every integral value of n, and as we may readily show for every fractional value of n also. We thus obtain as the result of applying an arbitrary function $\phi(D)$ to $e^{i\omega t}$,

$$\phi(D)e^{i\omega t} = \phi(i\omega)e^{i\omega t}. \tag{14}$$

We have for the nth derivative of the product of two functions the formula

$$D^n(u(t)v(t)) = \sum_{p=0}^{n} \frac{n!}{p!(n-p)!} u^{(p)}(t)v^{(n-p)}(t),$$

where $v^{(p)}(t) = D^{(p)}v(t)$ denotes the pth derivative.

We now introduce the operator

$$F(D) = \sum_{n=0,1,2,\ldots} a_n D^n$$

with a finite or infinite number of terms. We then have

$$F^{(p)}(D) = \sum_{n=p,\,p+1,\ldots} a_n \frac{n!}{(n-p)!} D^{n-p}. \tag{15}$$

If we apply $F(D)$ to the product $u(t)v(t)$, we obtain

$$F(D)(u(t)v(t)) = \sum_{n=0,1,\ldots} \sum_{p=0}^{n} a_n \frac{n!}{p!(n-p)!} u^{(p)}(t)v^{(n-p)}(t)$$

$$= \sum_{p=0}^{\infty} \frac{1}{p!} u^{(p)}(t) \sum_{n=p,\,p+1,\ldots} a_n \frac{n!}{(n-p)!} D^{n-p}v(t).$$

As a consequence of (15), we have

$$F(D)(u(t)v(t)) = \sum_{p=0}^{\infty} \frac{1}{p!} u^{(p)}(t)F^{(p)}(D)v(t).$$

If we omit $v(t)$ on both sides, we obtain the operational equation

$$F(D)u(t) = \sum_{p=0}^{\infty} \frac{1}{p!} u^{(p)}(t)F^{(p)}(D). \tag{16}$$

This well-known formula, which may be regarded as a generalization of that of Leibniz for the derivative of a product, permits us to interchange differentiations and multiplications.

We shall now consider the two operations

$$q_1 = u(t)F(D), \qquad q_2 = F(D)u(-t), \tag{17}$$

and compute their matrices. We make the simplifying assumption that the function $u(t)$ vanishes with all its derivatives for infinite arguments. We introduce the Fourier coefficients of $u(t)$:

$$U(\alpha) = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} e^{i\alpha t} u(t)dt.$$

By (12),

$$q_1(V, W) = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} e^{-\frac{2\pi i}{h} Vt} u(t)F(D)e^{\frac{2\pi i}{h} Wt} dt$$

$$= \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} u(t)F\left(\frac{2\pi i}{h} W\right) e^{\frac{2\pi i}{h}(W-V)t} dt,$$

or in other words

$$q_1(V, W) = F\left(\frac{2\pi i}{h} W\right) U\left(\frac{2\pi}{h}(W - V)\right). \tag{18}$$

Moreover,

$$q_2(V, W) = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} e^{-\frac{2\pi i}{h} Vt} F(D)u(-t)e^{\frac{2\pi i}{h} Wt} dt.$$

We here apply the formula (15) of Leibniz, and get

$$q_2(V, W) = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} e^{\frac{2\pi i}{h}(W-V)t} \sum_{p=0}^{\infty} \frac{1}{p!} u^{(p)}(-t)F^{(p)}\left(\frac{2\pi i}{h} W\right) dt.$$

By integration by parts, we convert the integrals

$$\int_{-\tau}^{\tau} u^{(p)}(t)e^{-\beta t} dt \qquad \left[\beta = \frac{2\pi i}{h}(V - W)\right]$$

which we find here into the form

$$\psi(t) + \beta^p \int_{-\tau}^{\tau} e^{-\beta t} u(t)dt.$$

It follows from our assumptions as to the behavior of $u(t)$ at infinity that $\psi(t)$ makes no contribution to our limiting expression.

Our lemma then yields

$$q_2(V, W) = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} e^{\frac{2\pi i}{h}(W-V)t} u(-t)dt$$
$$\sum_{p=0}^{\infty} \frac{1}{p!} F^{(p)}\left(\frac{2\pi i}{h} W\right) \left(\frac{2\pi i}{h}(V-W)\right)^p.$$

The sum involved in this expression is, however, $F\left(\frac{2\pi i}{h} V\right)$, and further

$$\lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} e^{\frac{2\pi i}{h}(W-V)t} u(-t)dt = U\left(\frac{2\pi}{h}(V-W)\right).$$

Hence

$$q_2(V, W) = F\left(\frac{2\pi i}{h} V\right) U\left(\frac{2\pi}{h}(V-W)\right). \tag{19}$$

By comparing (18) and (19), we see that q_2 is the " transposed " matrix of q_1, or that

$$q_1(W, V) = q_2(V, W), \qquad \tilde{q}_1 = q_2. \tag{20}$$

The condition for a matrix q_1 to be " Hermitian " then reads

$$\tilde{q}_1 = q_1^* \tag{21}$$

where the star symbolizes the operation of taking the conjugate complex quantity. Hence for the operator q_1 of (17), it reads

$$u(t)F(D) = F^*(D)u^*(-t). \tag{22}$$

More generally, for the operator

$$q = \sum_n u_n(t)F_n(D) \tag{23}$$

the *condition of Hermitianness* is

$$\sum_n u_n(t)F_n(D) = \sum_n F_n^*(D)u_n^*(-t). \tag{24}$$

We call an assemblage of functions *closed* with respect to the operations $q_1, q_2, \ldots, q_f$ under the following conditions: if $\phi(t)$ belongs to the set, so does $q_k\phi(t)$, and if $\phi_1(t)$ and $\phi_2(t)$ belong to the set, so does

$$c_1 \phi_1(t) + c_2\phi_2(t).$$

All the rules of matrix analysis, such as the definition of the

derivative of an operational or matrix function, remain valid for operators. We need not discuss them further.

§2. **The mechanics of quanta.** We represent a coördinate by a linear operator q instead of by a matrix. This has the advantage of enabling us to treat cases where the matrix representation goes to pieces, even though we introduce the notion of a continuous matrix. We must seek for the meaning of the statement that an atom is represented by a number of operators in the fact that an atom is a mechanism the characteristic oscillations of which are in some way given when the coördinate operators are determined.

Every coördinate has a corresponding momentum p, which is also an operator. The law of interchanging the product of a momentum and its coördinate is the quantum condition, and reads

$$pq-qp=\frac{h}{2\pi i}. \tag{25}$$

If we have two pairs of conjugate operators q_1, p_1, and q_2, p_2, then q_1 is permutable with q_2 and p_2, and p_1 is also permutable with q_2 and p_2.

We demand of all operators q and p:

(1) that they shall be Hermitian;

(2) that the functions to which we shall consider them applied shall form a closed set.

We further postulate that the laws of mechanics shall remain valid for the operators p and q, and more especially that the canonical equations

$$\dot{q}=\frac{\partial H(pq)}{\partial p}, \qquad \dot{p}=-\frac{\partial H(pq)}{\partial q} \tag{26}$$

shall hold for each pair q, p.

This gives us at once the law of the conservation of energy and the frequency condition. All we need to do is to replace the matrix formula $\dot{q}=\frac{2\pi i}{h}(Wq-qW)$ by the operational formula $\dot{q}=Dq-qD$. The result is that except for an arbitrary additive constant, the operator H is identical with the operator $\frac{h}{2\pi i}D$. H, therefore,

has the matrix

$$H(VW)=\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}e^{-\frac{2\pi i}{h}Vt}\,\frac{h}{2\pi i}\,De^{\frac{2\pi i}{h}Wt}\,dt$$

$$=W\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}e^{\frac{2\pi i}{h}(W-V)t}\,dt$$

$$=\begin{cases}0 & \text{for } V\neq W\\ W & \text{for } V=W\end{cases}$$

Furthermore, the laws of canonical transformations, the equation of Hamilton and Jacobi, and the whole theory of perturbations assume precisely the same form as in matrix theory. In particular, we may treat the case where the Hamiltonian function contains the time explicitly by taking $t=q_0$ as a new coördinate in $H(t,\ p_1q_1,\ \ldots,\ p_fq_f)$, introducing a corresponding momentum p_0, and working with the new Hamiltonian function

$$H^*=H(q_0,\ p_1q_1,\ \ldots,\ p_fq_f)+p_0. \tag{27}$$

We shall, however, not discuss these matters in more detail, but shall instead investigate the practical usefulness of our method in two instances. The first is the case of the harmonic oscillator, where we shall show how our formal theory necessarily brings us to the matrix representation and the well-known results of Heisenberg. The second is the case of uniform rectilinear motion, which cannot be treated by the direct use of matrices.

§3. **The harmonic oscillator.** The equation of motion

$$\ddot{q}+w_0^2q=0 \tag{28}$$

has the solution

$$q=e^{i\omega_0t}\phi_1(D)+e^{-i\omega_0t}\phi_2(D) \tag{29}$$

as follows from the relation

$$\dot{q}=Dq-qD=i\omega_0\left\{e^{i\omega_0t}\phi_1(D)-e^{-i\omega_0t}\phi_2(D)\right\}.$$

The transposed operator corresponding to q is

$$\tilde{q}=\phi_1(D)e^{-i\omega_0t}+\phi_2(D)e^{i\omega_0t} \tag{30}$$

in accordance with (17), and in conjunction with the formula (15) of Leibniz, gives us

$$\begin{aligned}\tilde{q} &= e^{-i\omega_0 t}\sum_{p=0}^{\infty}\frac{1}{p!}(-i\omega_0)^p\phi_1^{(p)}(D)+e^{i\omega_0 t}\sum_{p=0}^{\infty}\frac{1}{p!}(i\omega_0)^p\phi_2^{(p)}(D)\\ &= e^{-i\omega_0 t}\phi_1(D-i\omega_0)+e^{i\omega_0 t}\phi_2(D+i\omega_0).\end{aligned}$$

The condition (24) that q is Hermitian hence gives us

$$\phi_2^*(D+i\omega_0)=\phi_1(D). \tag{31}$$

Similarly, if we put $p=\mu\dot{q}$, the quantum condition (25) leads to

$$\begin{aligned}\frac{h}{2\pi i\mu} &= 2i\omega_0\left\{e^{i\omega_0 t}\phi_1(D)e^{-i\omega_0 t}\phi_2(D)-e^{-i\omega_0 t}\phi_2(D)e^{i\omega_0 t}\phi_1(D)\right\}\\ &= 2i\omega_0\left\{\sum_{p=0}^{\infty}\frac{1}{p!}(-i\omega_0)^p\phi_1^{(p)}(D)\phi_2(D)-\sum_{p=0}^{\infty}\frac{1}{p!}(i\omega_0)^p\phi_2^{(p)}(D)\phi_1(D)\right\}\\ &= 2i\omega_0(\phi_1(D-i\omega_0)\phi_2(D)-\phi_2(D+i\omega_0)\phi_1(D)).\end{aligned}$$

Combining this with (31), we get

$$\frac{h}{4\pi\mu\omega_0}=|\phi_1(D)|^2\cdot|\phi_1(D-i\omega_0)|^2. \tag{32}$$

This is a difference equation in the operator $\phi_1(D)$. By applying it to the function $e^{\frac{2\pi i}{h}Wt}$, we turn it into an ordinary difference equation in ϕ_1. Hence by (14)

$$\frac{h}{4\pi\mu\omega_0}=\left|\phi_1\left(\frac{2\pi i}{h}W\right)\right|^2-\left|\phi_1\left(\frac{2\pi i}{h}W-i\omega_0\right)\right|^2. \tag{33}$$

With the use of (30), we get

$$\left|\phi_2\left(\frac{2\pi i}{h}W+i\omega_0\right)\right|^2=\left|\phi_1\left(\frac{2\pi i}{h}W\right)\right|^2. \tag{34}$$

The difference equation (33) cannot be satisfied for all values of W, as this would lead to negative values of $|\phi_1|^2$. We must therefore select a set of possible values of W. As the precise value of W itself is of no importance, we may put 0 for the first value of W for which $\phi_1\left(\frac{2\pi i}{h}W\right)$ is different from zero. Our equations

then show that the values of W which we must consider are the following:

$$W_k = h\frac{\omega_0}{2\pi}k \begin{cases} k=0,\ 1,\ 2,\ \ldots \text{ for } \phi_1 \\ k=1,\ 2,\ \ldots \quad \text{for } \phi_2 \end{cases}. \tag{35}$$

We shall finally have to show that the functions involving only the frequencies $\nu(m, n) = \frac{1}{h}(W_m - W_n) = \frac{\omega_0}{2\pi}(m-n)$ form a closed set.

We get

$$\left|\phi_1\left(\frac{2\pi i}{h}W_k\right)\right|^2 = \left|\phi_2\left(\frac{2\pi i}{h}W_{k+1}\right)\right|^2 = \frac{h}{4\pi\mu\omega_0}(k+1). \tag{36}$$

The theory determines only the moduli of ϕ_1 and ϕ_2, not their phases. We put

$$\begin{aligned} \phi_1 &= C\sqrt{k+1}\, e^{i\delta_{k+1}} \\ \phi_2 &= C\sqrt{k}\, e^{-i\delta_k} \\ C &= \frac{h}{4\pi\mu\omega_0}. \end{aligned} \tag{37}$$

The operator $\phi(D)$ is discontinuous, and corresponds to an actual matrix. We form the sum of a line in accordance with (10), and get

$$\begin{aligned} q(W_k, t) &= e^{-\frac{2\pi i}{h}W_k t}\left(e^{i\omega_0 t}\phi_1\left(\frac{2\pi i}{h}W_k\right) + e^{-i\omega_0 t}\phi_2\left(\frac{2\pi i}{h}W_k\right)\right)e^{\frac{2\pi i}{h}W_k t} \\ &= C\sqrt{k+1}\, e^{i(\omega_0 t+\delta_{k+1})} + C\sqrt{k}\, e^{-i(\omega_0 t+\delta_k)}. \end{aligned} \tag{38}$$

By (12), we get the matrix by the process of taking the mean as follows:

$$q_{kl} = q(W_l, W_k) = \lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau} q(W_k, t)e^{-\frac{2\pi i}{h}(W_l - W_k)t}\,dt.$$

If we here substitute for $q(W_k, t)$ its value, we see that all terms vanish except when

$$\pm\omega_0 = \frac{2\pi}{h}(W_l - W_k), \quad i.e. W_l = W_k \pm \frac{h}{2\pi}\omega_0 = \frac{h\omega_0}{2\pi}(k\pm 1), \quad l = k\pm 1.$$

and we thus find

$$q_{kl} = C\sqrt{k+1}\,\delta_{k+1,l} + C\sqrt{k}\,\delta_{k-1,l}. \tag{39}$$

This confirms Heisenberg's result, and further shows that our calculus automatically yields the discontinuous matrix as the only solution, for since the application of the operation q to $e^{i\omega t}$ gives us no other frequencies than $\omega \pm \omega_0$, the functions containing only these frequencies form a closed set.

The energy function

$$H = \frac{1}{2\mu} p^2 + \frac{\mu\omega_0^2}{2} q^2$$

turns out to be nothing but the diagonal matrix with the elements

$$H_{kk} = \frac{h\omega_0}{2\pi}\left(k + \frac{1}{2}\right) \qquad (k = 0, 1, 2, \ldots) \tag{40}$$

which is in complete accordance with the values we have obtained in (35).

§4. **Unaccelerated motion in one dimension.** The equation of motion

$$\ddot{q} = 0 \tag{41}$$

has the solution

$$q = t\phi_1(D) + \phi_2(D), \tag{42}$$

inasmuch as the derivative of this expression is

$$\dot{q} = Dq - qD = \phi_1(D), \tag{43}$$

which has a zero derivative.

We have in accordance with (24)

$$\tilde{q} = \phi_1(D)t + \phi_2(D).$$

Combining this with Leibniz' formula (16), we get

$$\tilde{q} = t\phi_1(D) + \phi_1'(D) + \phi_2(D). \tag{44}$$

The condition of Hermitianness (24) then gives us

$$\begin{cases} \phi_1(D) = \phi_1^*(D) \\ \phi_2(D) = [\phi_1'(D) + \phi_2(D)]^* \end{cases} \tag{45}$$

The quantum condition (25) here reads

$$\frac{h}{2\pi i\mu} = \phi_1(D)[t\phi_1(D)+\phi_2(D)]-[t\phi_1(D)+\phi_2(D)]\phi_1(D),$$

or by (15)

$$\frac{h}{2\pi i\mu} = \phi_1'(D)\phi_1(D). \tag{46}$$

If we apply this operator to $e^{\frac{2\pi i}{h}Wt}$, we get

$$\frac{h}{2\pi i\mu} = \phi_1(u)\phi_1'(u), \quad u=\frac{2\pi i}{h}W.$$

This is a differential equation for u, with the solution

$$\phi_1(u) = \sqrt{\frac{hu}{\pi i\mu}}. \tag{47}$$

This determines $\phi_1(D)$. To find $\phi_2(D)$, we apply it to e^{iut} and use (45). It follows from

$$\phi_2(u) = \phi^{*\prime}(u)+\phi^*{}_2(u)$$

that the imaginary part of $\phi_2(u)$ is equal to $-\frac{1}{2}\phi_1'(u)$, so that

$$\phi_2(u) = \psi(u)-\tfrac{1}{2}\phi_1'(u)$$

where $\psi(u)$ is real. If we substitute the value of ϕ_1, obtained in (47), we obtain

$$\phi_2(u) = \psi(u) = \frac{1}{4}\sqrt{\frac{h}{\pi i\mu u}}. \tag{48}$$

Hence

$$q = t\sqrt{\frac{hD}{\pi i\mu}} - \frac{1}{4}\sqrt{\frac{h}{\pi i\mu D}}+\psi(D). \tag{49}$$

By (11), the sum of the terms in a row is

$$q(t,W) = e^{-\frac{2\pi i}{h}Wt}\left(t\sqrt{\frac{hD}{\pi i\mu}}\right) - \frac{1}{4}\sqrt{\frac{hD}{\pi i\mu D}}+\psi(D)e^{\frac{2\pi i}{h}Wt}$$

and hence by (14),

$$q(t,\ W) = t\sqrt{\frac{2W}{\mu}}+\frac{ih}{4\pi\sqrt{\mu W}}+\psi\left(\frac{2\pi i}{h}W\right). \tag{50}$$

If W is large, $\frac{h}{4\pi\sqrt{\mu W}}$ is small. Hence we have asymptotically

$$q(t, W) = t\sqrt{\frac{2W}{\mu}} + \psi\left(\frac{(2\pi i W}{h}\right).$$

This is in complete agreement with the classical formula for unaccelerated motion, for we have

$$p(t, W) = \mu\sqrt{\frac{2W}{\mu}}, \qquad W = \frac{p^2}{2\mu}. \tag{51}$$

In this case, the sum of the terms in a row really represents the motion.

The matrix is, however, degenerate, as is obvious from definition (12). The motion is entirely free from periodic components. It is nevertheless just as amenable to our methods as a periodic motion.

Our representation is as close to the classical theory as seems to be consistent with the foundations of the quantum theory.

THE HARMONIC ANALYSIS OF IRREGULAR MOTION

By Norbert Wiener

The author has already developed the theory of the harmonic analysis of a very general class of functions which neither have properties of periodicity, as do those to which the method of Fourier series applies, nor of vanishing at infinity, as is essential for the application of the classical forms of the Fourier integral.[1] The notions there employed are closely akin to those which had been previously introduced by Hahn,[2] although Hahn's discussion is strictly confined to ordinary convergence and to Stieltjes integrals of the classical sort, whereas the present author is primarily interested in convergence in the mean and in theorems of the Hurwitz and Riesz-Fischer order, and has found the classical Stieltjes integral inadequate to his needs. In a recent paper presented to the Academy of Sciences at Vienna,[3] of which Professor Hahn has been kind enough to send me a proof-sheet, integrals of the form

$$\lim_{\mu\to\infty}\frac{1}{\mu\pi}\int_0^{\mu}d\lambda\int_0^{\lambda}\left(\cos\mu x\frac{d^2\Phi_2(\tau)}{d\tau}+\sin\mu x\frac{d^2\Psi_2(\tau)}{d\tau}\right)$$

are introduced, where

$$\Phi_2(\mu)=2\int_{-\infty}^{\infty}f(x)\frac{\sin^2\frac{\mu}{2}x}{x^2}dx;\ \Psi_2(\mu)=\int_{-1}^{1}f(x)\frac{\mu x-\sin\mu x}{x^2}dx$$
$$-\int_1^{\infty}f(x)\frac{\sin\mu x}{x^2}dx-\int_{-\infty}^{-1}f(x)\frac{\sin\mu x}{x^2}dx.$$

The present author has been compelled in the course of his own researches to make use of integrals of similar form. It therefore seems to him an appropriate time to communicate the results of these researches.

The theory developed by the author in his *Zeitschrift* paper has one important defect. While the theory embodied in that paper

[1] Mathematische Zeitschrift, Bd. 24, Heft 3, pp. 575–616.

[2] H. Hahn: Über Fouriersche Reihen und Integrale, Bericht über die Jahresversammlung zu Innsbruck, Jahresbericht der deutschen Mathematiker-vereinigung, 33 (1925), p. 107.

[3] Jahrgang 1925, Nr. 25, Sitzung vom 3. Dezember, 1925.

has at least a formal niche for functions with continuous spectra, in all the more detailed developments of that paper, a certain hypothesis of uniformity plays an important rôle. This is that the functions discussed should be well-behaved. To the present time, the author has been unable either to produce a single example of a well-behaved function with a continuous spectrum, or to demonstrate that no such functions exist. In the theory here expounded, the necessary existence proofs may be filled in with the aid of the theory of probabilities, aided, it is true, by the Zermelo principle of selection. These existence proofs, however, will be reserved for a later communication.

Included in the range of ideas treated in this paper are the theory of white light, of noise, of the periodogram, and of the so-called Schrotteffekt. Inasmuch, however, as probability considerations play a large part in most of these topics, the author will put off a detailed discussion of them to a later paper.

§1. The Coefficient-Functions of a Trigonometrical Development.

Let $f(t)$ be a summable function defined for all real arguments, and let

$$\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}[f(t)]^2dt = M\left\{[f(t)]^2\right\}$$

exist. Then

$$\int_1^\infty \frac{dA}{A^2}\int_{-A}^{A}[f(t)]^2dt = \int_{A=1.}^{A=\infty} d\left(-\frac{1}{2A^2}\right)\int_{-A}^{A}[f(t)]^2dt$$

$$= \frac{1}{2}\int_{-1}^{1}[f(t)]^2dt + \int_1^\infty \frac{[f(A)]^2+[f(-A)]^2}{2A^2}\,dA \qquad (1)$$

exists. Now,

$$\left[\int_{B_1} f(t)\,\frac{1-\cos ut}{t^2}\,dt\right]^2 \leq 4\int_{B_1}\frac{[f(t)]^2}{t^2}\,dt\int_{B_1}\frac{dt}{t^2}.$$

$$\leq\left[\frac{4}{B_1}-\frac{4}{B_2}\right]\int_{B_1}\frac{[f(t)]^2}{t^2}.$$

It follows that

$$\Gamma(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{1-\cos ut}{t^2}\,dt \qquad (2)$$

exists, and indeed exists as a limit uniform in u. A similar proof will show the existence of

$$\Delta(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{ute^{-t^2} - \sin ut}{t^2}\,dt \tag{3}$$

uniformly in u.

We shall here introduce the following

Lemma. *Let $g(x)$ be a function summable and of summable square over the range $(-a, a)$. Let*

$$\frac{1}{4h^2}\int_{-a}^{a}[g(x+h)-g(x-h)]^2dx$$

be bounded as h approaches 0. Then there is a function $g'(x)$ such that

$$\int_{-a}^{a}\left[g(x)-g(0)-\int_0^x g'(y)dy\right]^2 = 0. \tag{4}$$

To prove this, let us expand $g(x)$ in the Fourier series

$$g(x) = \frac{a_0}{2} + \sum_1^\infty\left[a_n\cos\frac{n\pi x}{a} + b_n\sin\frac{n\pi x}{a}\right].$$

We shall have

$$\frac{1}{4h^2}\int_{-a}^{a}[g(x+h)-g(x-h)]^2dx = \frac{a}{h^2}\sum_1^\infty[a_n^2+b_n^2]\sin^2\frac{n\pi h}{a}.$$

Hence

$$\sum_1^m n^2[a_n^2+b_n^2]$$

is bounded, and by the Riesz-Fischer theorem, there is a function to which

$$\frac{\pi}{a}\sum_1^\infty n\left[b_n\cos\frac{n\pi x}{a} - a_n\sin\frac{n\pi x}{a}\right]$$

converges in the mean. This fulfills the requirements for $g'(x)$.

It may be proved without difficulty that $\Gamma'(u)$ and $\Delta'(u)$, defined in this sense, exist over any finite range of u. It is readily seen that the definitions for two overlapping ranges are then con-

sistent. To prove the existence of $\Gamma'(u)$, we have only to note that

$$\frac{1}{4h^2}\int_{-a}^{a} du \left\{ \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{1-\cos(u+h)t}{t^2}\,dt \right.$$
$$\left. -\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{1-\cos(u-h)t}{t^2}\,dt \right\}^2 dx$$
$$=\frac{1}{4h^2}\int_{-a}^{a} du \left\{ \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{2\sin ut \sin ht}{t^2}\,dt \right\}^2$$
$$\leq \frac{1}{2h^2}\int_{-\infty}^{\infty} du \left\{ \frac{1}{\pi}\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right] f(t)\,\frac{2\sin ut \sin ht}{t^2}\,dt \right\}^2$$
$$+\frac{1}{2h^2}\int_{-a}^{a} du \left\{ \frac{1}{\pi}\int_{-A}^{A} f(t)\,\frac{2\sin ut \sin ht}{t^2}\,dt \right\}^2$$
$$\leq \frac{1}{\pi h^2}\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right] [f(t)]^2\,\frac{\sin^2 ht}{t^4}\,dt$$
$$+\frac{1}{2}\int_{-a}^{a} du \left\{ \frac{1}{\pi}\int_{-A}^{A} f(t)\,\frac{2\sin ut}{t}\,dt \right\}^2$$
$$\leq \frac{1}{2\pi}\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right]\frac{[f(t)]^2}{t^2}\,dt+\frac{1}{2}\int_{-a}^{a} du \left\{ \frac{1}{\pi}\int_{-A}^{A} f(t)\,\frac{2\sin ut}{t}\,dt \right\}^2.$$

As this is independent of h, the existence of $\Gamma'(u)$ follows at once. A similar proof demonstrates the existence of $\Delta(u)$.

We have

$$\frac{1}{h^3}\int_{-\infty}^{\infty} [f(t)]^2\,\frac{\sin^4 ht}{t^4}\,dt = \frac{1}{h^3}\int_{t=0}^{t=\infty} \frac{\sin^4 ht}{t^4}\,d\int_{-t}^{t} [f(\xi)]^2 d\xi$$
$$= -\frac{1}{h^3}\int_{0}^{\infty} \left\{ \int_{-t}^{t} [f(\xi)]^2 d\xi \right\} \frac{d}{dt}\left[\frac{\sin^4 ht}{t^4}\right] dt$$
$$= -\frac{1}{h^3}\int_{0}^{\infty} \left\{ \frac{1}{2t}\int_{-t}^{t} [f(\xi)]^2 d\xi \right\} \frac{8\sin^3 ht(ht\cos ht-\sin ht)}{t^4}\,dt. \tag{5}$$

It may be verified immediately that

$$\int_{0}^{\infty} \frac{8\sin^3 x(x\cos x-\sin x)}{x^4}\,dx = -\int_{-\infty}^{\infty} \frac{\sin^4 u}{u^4}\,du = -\frac{2\pi}{3} \tag{6}$$

converges absolutely. Hence

$$\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}[f(t)]^2\frac{\sin^4 ht}{t^4}\,dt=\frac{2\pi}{3}\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}[f(\xi)]^2 d\xi. \tag{7}$$

It follows at once from the properties of the Fourier transformation that

$$\begin{aligned}
&\frac{1}{h^3}\int_{-\infty}^{\infty}[f(t)]^2\frac{\sin^4 ht}{t^4}\,dt\\
&\quad=\frac{2}{\pi h^3}\int_0^{\infty}\left\{\left[\int_{-\infty}^{\infty}f(t)\frac{\sin^2 ht\sin tu}{t^2}\,dt\right]^2\right.\\
&\qquad\left.+\left[\int_{-\infty}^{\infty}f(t)\frac{\sin^2 ht\cos tu}{t^2}\,dt\right]^2\right\}du\\
&\quad=\frac{2}{\pi h^3}\int_0^{\infty}\left\{\left[\int_{-\infty}^{\infty}f(t)\frac{(1-\cos 2ht)\sin tu}{2t^2}\,dt\right]^2\right.\\
&\qquad\left.+\left[\int_{-\infty}^{\infty}f(t)\frac{(1-\cos 2\,ht)\cos tu}{2t^2}\,dt\right]^2\right\}du\\
&\quad=\frac{2}{\pi h^3}\int_0^{\infty}\left\{\left[\int_{-\infty}^{\infty}f(t)\frac{\sin(t+2h)u+\sin(t-2h)u-2\sin tu}{4t^2}\,dt\right]^2\right.\\
&\qquad\left.+\left[\int_{-\infty}^{\infty}f(t)\frac{\cos(t+2h)u+\cos(t-2h)u-2\cos tu}{4t^2}\,dt\right]^2\right\}du\\
&\quad=\frac{\pi}{8h^3}\int_0^{\infty}\left\{[\Delta(u+2h)+\Delta(u-2h)-2\Delta(u)]^2+[\Gamma(u+2h)\right.\\
&\qquad\left.+\Gamma(u-2h)-2\Gamma(u)]^2\right\}du.
\end{aligned}\tag{8}$$

As an immediate consequence, we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon^3}\int_0^{\infty}\left\{[\Delta(u+\epsilon)+\Delta(u-\epsilon)-2\Delta(u)]^2\right.\\
&\qquad\left.+[\Gamma(u+\epsilon)+\Gamma(u-\epsilon)-2\Gamma(u)]^2\right\}du\\
&\quad=\frac{2}{3}\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}[f(\xi)]^2 d\xi=\frac{2}{3}M\{[f(\xi)]^2\},
\end{aligned}\tag{9}$$

in the sense that the former of these two limits exists whenever the latter does, and is identical with it.

There is another interesting property of $\Gamma(u)$ and $\Delta(u)$. Let us interpret

$$\int_0^A \cos ut\, d\Gamma'(u)+\int^A \sin ut\, d\Delta'(u)$$

as the result which we obtain by integrating by parts as indicated —that is, as

$$\begin{aligned}&\cos At\Gamma'(A)+\sin At\,\Delta'(A)+t\int_0^A \sin ut\, d\Gamma(u)-t\int_0^A \cos ut\, d\,\Delta(u)\\&=\cos At\Gamma'(A)+\sin At\,\Delta'(A)+t[\sin At\Gamma(A)-\cos At\,\Delta(A)]\\&-t^2\Big[\int_0^A \cos ut\,\Gamma(u)du+\int_0^A \sin ut\,\Delta(u)du\Big].\end{aligned}\tag{10}$$

We then have

$$\begin{aligned}&\int_0^A \cos ut\, d\Gamma'(u)+\int_0^A \sin ut\, d\,\Delta'(u)\\&=\operatorname*{limit\ in\ mean}_{h\to 0}\left\{\frac{1}{\pi h}\int_{-\infty}^{\infty}f(\tau)\,\frac{\cos At\sin A\tau\sin h\tau}{\tau^2}\,d\tau\right.\\&\qquad\left.-\frac{1}{\pi h}\int_{-\infty}^{\infty}f(\tau)\frac{\sin At[\cos A\tau\sin h\tau-h\tau e^{-\tau^2}]}{\tau^2}\,d\tau\right\}\\&+t\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}f(\tau)\,\frac{\sin At(1-\cos A\tau)}{\tau^2}d\tau\right.\\&\qquad\left.-\frac{1}{\pi}\int_{-\infty}^{\infty}f(\tau)\,\frac{\cos At(A\tau e^{-\tau^2}-\sin A\tau)}{\tau^2}d\tau\right\}\\&-t^2\int_0^A du\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}f(\tau)\,\frac{\cos ut\,(1-\cos u\tau)}{\tau^2}\,d\tau\right.\\&\qquad\left.+\frac{1}{\pi}\int_{-\infty}^{\infty}f(\tau)\,\frac{\sin ut\,(u\tau e^{-\tau^2}-\sin u\tau)}{\tau^2}\,d\tau\right\}.\end{aligned}\tag{11}$$

It is easy to verify that, in the last integral, the conditions for

integrating under the sign of integration are fulfilled, and that

$$t^2\int_0^A du\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)\frac{\cos ut(1-\cos u\tau)}{\tau^2}d\tau\right.$$

$$\left.+\frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)\frac{\sin ut(u\tau e^{-\tau^2}-\sin u\tau)}{\tau^2}d\tau\right\}$$

$$=\frac{t}{\pi}\int_{-\infty}^{\infty} f(\tau)\frac{\sin At}{\tau^2}\,d\tau-\frac{t^2}{2\pi}\int_{-\infty}^{\infty} f(\tau)\frac{\sin A(t+\tau)}{\tau^2(t+\tau)}d\tau$$

$$-\frac{t^2}{2\pi}\int_{-\infty}^{\infty} f(\tau)\frac{\sin A(\tau-t)}{\tau^2(\tau+t)}\,d\tau$$

$$+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(\tau)\sin At\,\tau e^{-\tau^2}}{\tau^2}\,d\tau-\frac{t}{\pi}\int_{-\infty}^{\infty}\frac{Af(\tau)\cos At\,\tau e^{-\tau^2}}{\tau^2}\,d\tau$$

$$+\frac{t^2}{2\pi}\int_{-\infty}^{\infty} f(\tau)\frac{\sin A(t+\tau)}{\tau^2(t+\tau)}d\tau-\frac{t^2}{2\pi}\int_{-\infty}^{\infty} f(\tau)\frac{\sin A(\tau-t)}{\tau^2(\tau-t)}d\tau.$$

Hence

$$\int_0^A \cos ut\,d\Gamma'(u)+\int_0^A \sin ut\,d\Delta'(u)$$

$$=\lim_{h\to 0}\text{l. i. m.}\frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)\frac{\sin A(\tau-t)}{\tau^2}\left[\frac{\sin h\tau}{h}+\frac{\tau t}{\tau-t}\right]d\tau$$

$$=\frac{1}{\pi}\int_{-B}^{B} f(\tau)\frac{\sin A(\tau-t)}{\tau-t}d\tau+\frac{1}{\pi}\left[\int_{-\infty}^{-B}+\int_{B}^{\infty}\right]\frac{t\sin A(\tau-t)}{\tau(\tau-t)}\,d\tau$$

$$+\lim_{h\to 0}\text{l. i. m.}\frac{1}{\pi}\left[\int_{-\infty}^{-B}+\int_{B}^{\infty}\right] f(\tau)\frac{\sin A(\tau-t)\sin h\tau}{h\tau^2}\,d\tau. \qquad (12)$$

Now let us consider only such values of t as lie in the range $(-B+a,\ B-a)$. We have

$$\lim_{A\to\infty}\text{l. i. m.}\frac{1}{\pi}\int_{-B}^{B} f(\tau)\frac{\sin A(\tau-t)}{\tau-t}\,d\tau=f(t), \qquad (13)$$

and

$$\lim_{A\to\infty}\frac{1}{\pi}\left[\int_{-\infty}^{-B}+\int_{B}^{\infty}\right] f(\tau)\frac{t\sin A(\tau-t)}{\tau(\tau-t)}\,d\tau=0 \qquad (14)$$

uniformly in t over any finite range. By the well-known theorem

of Plancherel on the Fourier transformation, we know that

$$\frac{1}{\pi}\left[\int_{-C}^{-B}+\int_{B}^{C}\right]f(\tau)\frac{\sin A\ (\tau-t)}{\tau}\,d\tau$$

converges in the mean in A and t as C becomes infinite to some function $G(A, t)$, since

$$\left[\int_{-\infty}^{-B}+\int_{B}^{\infty}\right]\frac{|f(\tau)|^2}{\tau^2}\,d\tau$$

is bounded. Let us put

$$\begin{aligned}&\frac{1}{\pi}\left[\int_{-\infty}^{-B}+\int_{B}^{\infty}\right]f(\tau)\frac{\sin A(\tau-t)\sin h\tau}{h\tau^2}\,d\tau\\&=\frac{1}{\pi}\left[\int_{-\infty}^{-C}+\int_{C}^{\infty}\right]f(\tau)\frac{\sin A(\tau-t)\sin h\tau}{h\tau^2}\,d\tau\\&+\frac{1}{\pi}\left[\int_{-C}^{-B}+\int_{B}^{C}\right]f(\tau)\frac{\sin A(\tau-t)\sin h\tau}{h\tau^2}\,d\tau.\end{aligned}$$

It is easy to see that

$$\begin{aligned}&\lim_{h\to 0}\frac{1}{\pi}\left[\int_{-C}^{-B}+\int_{B}^{C}\right]f(\tau)\frac{\sin A(\tau-t)\sin h\tau}{h\tau^2}\,d\tau\\&=\frac{1}{\pi}\left[\int_{-C}^{-B}+\int_{B}^{C}\right]f(\tau)\frac{\sin A(\tau-t)}{\tau}\,d\tau\end{aligned}$$

uniformly over any range of A and t. Moreover,

$$\begin{aligned}&\int_0^a\left\{\frac{1}{\pi}\left[\int_{-\infty}^{-C}+\int_{C}^{\infty}\right]f(\tau)\frac{\sin A(\tau-t)\sin h\tau}{h\tau^2}\,d\tau\right\}^2 dA,\\&=\left|\frac{1}{\pi^2}\left[\int_{\infty}^{-C}+\int_{C}^{\infty}\right]f(\tau)\frac{\sin h\tau}{h\tau^2}\,d\tau\left[\int_{-\infty}^{-C}+\int_{C}^{\infty}\right]f(\sigma)\frac{\sin h\sigma}{h\sigma^2}\,d\sigma\right.\\&\left.\cdot\frac{1}{2}\left[\frac{\sin a(\tau-\sigma)}{\tau-\sigma}-\frac{\sin a(\tau+\sigma-2t)}{\tau+\sigma-2t}\right]\right|\\&\leq\frac{1}{\pi^2}\left[\int_{-\infty}^{-C}+\int_{C}^{\infty}\right]d\tau\left[\int_{-\infty}^{-C}+\int_{C}^{\infty}\right]\left|\frac{f(\tau)f(\sigma)}{\tau\sigma}\right|\\&\left[\left|\frac{\sin a(\tau-\sigma)}{\tau-\sigma}\right|+\left|\frac{\sin a(\tau+\sigma-2t)}{\tau+\sigma-2t}\right|\right]d\sigma\end{aligned}$$

if the latter integral exists. This will be the case when

$$\left[\int_{-\infty}^{-C}+\int_{C}^{\infty}\right]\frac{[f(\tau)]^2}{\tau^{4/3}}\,d\tau;$$

$$\left[\int_{\infty}^{-C}+\int_{C}^{\infty}\right]d\tau\left[\int_{-\infty}^{-C}+\int_{C}^{\infty}\right]\frac{\sin^2\alpha(\tau-\sigma)}{(\tau-\sigma)^2(\tau\sigma)^{2/3}}d\sigma;$$

$$\left[\int_{\infty}^{-C}+\int_{C}^{\infty}\right]d\tau\left[\int_{-\infty}^{-C}+\int_{C}^{\infty}\right]\frac{\sin^2\alpha(\tau+\sigma-2t)}{(\tau+\sigma-2t)^2(\tau\sigma)^{2/3}}d\sigma$$

all exist. The existence of the first of these integrals follows from the existence of

$$\lim_{A\to\infty}\frac{1}{2A}\int_{A}^{A}[f(t)]^2dt$$

by the fact that

$$\int_1^\infty\frac{dA}{A^{7/3}}\int_A^A[f(t)]^2\,dt=\int_{A=1}^{A=\infty}d\left(\frac{-3}{4A^{4/3}}\right)\int_{-A}^{A}[f(t)]^2dt$$

$$=\frac{3}{4}\int_{-1}^{1}[f(t)]^2+\frac{3}{4}\int_1^\infty\left\{[f(A)]^2+[f(-A)]\right\}^2A^{-4/3}dA.$$

As to the second integral, we have

$$\left|\int_C^{C_1}d\tau\int_C^{C_1}\frac{\sin^2\alpha(\tau-\sigma)}{(\tau-\sigma)^2(\tau\sigma)^{2/3}}d\sigma\right|\leq\frac{1}{2}\int_0^{2C_1}\frac{\sin^2\alpha v}{v^2}dv\int_{v+2C}^{2C_1}\frac{du}{(u^2-v^2)^{2/3}}$$

$$+\frac{1}{2}\int_{-2C_1}^{0}\frac{\sin^2\alpha v}{v^2}dv\int_{2C-v}^{2C_1}\frac{du}{(u^2-v^2)^{2/3}}$$

$$\leq\int_0^\infty\frac{\sin^2\alpha v}{v^2}dv\int_{2C}^{\infty}\frac{dw}{w^2}.$$

The existence of the other parts of the second integral, as well as the existence of the third integral, and the fact that it has for any finite range of t an upper bound vanishing as $c\to\infty$, may be proved in the same manner.

We may readily conclude that

$$\text{l. i. m.}_{C\to\infty}\frac{1}{\pi}\left[\int_{-\infty}^{-C}+\int_{C}^{\infty}\right]f(\tau)\,\frac{\sin A(\tau-t)\sin h\tau}{h\tau^2}\,d\tau=0$$

in the variable A uniformly in t and h. Hence

$$\text{l. i. m.}_{C\to\infty}\ \text{l. i. m.}_{h\to 0}\ \frac{1}{\pi}\left[\int_{-\infty}^{-C}+\int_{C}^{\infty}\right] f(\tau)\,\frac{\sin A(\tau-t)\,\sin h\tau}{h\tau^2}\,d\tau=0.$$

Therefore

$$\text{l. i. m.}_{C\to\infty}\ \text{l. i. m.}_{h\to 0}\ \frac{1}{\pi}\left[\int_{-C}^{-B}+\int_{B}^{C}\right] f(\tau)\,\frac{\sin A(\tau-t)\,\sin h\tau}{h\tau^2}\,d\tau$$
$$=\text{l. i. m.}_{h\to 0}\,\frac{1}{\pi}\left[\int_{-\infty}^{-B}+\int_{B}^{\infty}\right] f(\tau)\,\frac{\sin A(\tau-t)\,\sin h\tau}{h\tau^2}\,d\tau.$$

That is,

$$\text{l. i. m.}_{C\to\infty}\,\frac{1}{\pi}\left[\int_{-C}^{-B}+\int^{C}\right] f(\tau)\,\frac{\sin A(\tau-t)}{\tau}\,d\tau$$
$$=\text{l. i. m.}_{h\to 0}\ \frac{1}{\pi}\left[\int_{-\infty}^{-B}+\int_{B}^{\infty}\right] f(\tau)\,\frac{\sin A(\tau-t)\,\sin h\tau}{h\tau^2}\,d\tau.$$

By the use of a now familiar theorem concerning Fourier transforms,

$$\int_{-\infty}^{\infty}\left\{\frac{1}{\pi}\left[\int_{-C}^{-B}+\int_{B}^{C}\right] f(\tau)\,\frac{\sin A(\tau-t)}{\tau}\,d\tau\right\}^2 dA$$
$$=\frac{1}{\pi}\left[\int_{-C}^{-B}+\int_{B}^{C}\right]\frac{[f(u+t)]^2}{[u+t)]^2}\,du. \tag{15}$$

Hence

$$\int_{-\infty}^{\infty}\left\{\text{l. i. m.}_{h\to\rho}\,\frac{1}{\pi}\left[\int_{-\infty}^{-B}+\int_{B}^{\infty}\right] f(\tau)\,\frac{\sin A(\tau-t)\,\sin h\tau}{h\tau^2}\,d\tau\right\}^2 dA$$
$$=\frac{1}{\pi}\left[\int_{-\infty}^{-B}+\int_{B}^{\infty}\right]\frac{[f(u+t)]^2}{[u+t]^2}\,du. \tag{16}$$

Since we know that this latter integral exists, we have

$$\lim_{A\to\infty}\int_{A}^{A+1} du\left\{\text{l.i.m.}_{h\to 0}\,\frac{1}{\pi}\left[\int_{-\infty}^{-B}+\int_{B}^{\infty}\right] f(\tau)\,\frac{\sin u(\tau-t)\,\sin h\tau}{h\tau^2}\,d\tau\right\}^2 = \quad . \tag{17}$$

If we combine this with propositions (12), (13), and (14), we see that

$$\lim_{A\to\infty}\int_A^{A+1}\left\{f(t)-\int_0^{\infty}\cos ut\,d\Gamma'(u)-\int_0^{\infty}\sin ut\,d\Delta'(u)\right\}^2 dw=0. \tag{18}$$

In the language of the author's previous paper,

$$\int_0^A \cos ut\,d\Gamma'(u)+\int_0^A \sin ut\,d\Delta'(u)$$

converges almost in the mean to $f(t)$ as A becomes infinite. We thus see that there is a certain reciprocality in the relation between $f(t)$ and the pair of functions $\Gamma'(u)$ and $\Delta'(u)$, inasmuch as each is enough to determine the other except at a set of points of measure 0.

As the ideas of this section are a little hard to picture, it may be worth while to illustrate them by a simple example. Let

$$f(t)=\sum_1^m(a_n\cos\alpha_n t+b_n\sin\alpha_n t) \qquad (\text{no } \alpha_n=0).$$

Then

$$\Gamma'(u)=\sum_{\alpha_n<u}a_n;$$

$$\Delta'(u)=\sum_{\alpha_n<u}b_n+C_1;$$

$$\Gamma(u)=\sum_{\alpha_n<u}a_n(u-\alpha_n)$$

$$\Delta(u)=\sum_{\alpha_n<u}b_n(u-\alpha_n)+C_1u+C_2.$$

If ϵ is sufficiently small, we have in the neighborhood of α_n

$$\begin{aligned}\Gamma(u+\epsilon)+\Gamma(u-\epsilon)-2\Gamma(u)&=a_n(u+\epsilon-\alpha_n); && [\alpha_n-\epsilon<u<\alpha_n]\\ &=a_n(\epsilon-u+\alpha_n); && [\alpha_n<u<\alpha_n+\epsilon]\\ &=0. && [\,|u-\alpha_n|>\epsilon\,]\end{aligned}$$

Hence

$$\frac{1}{\epsilon^3}\int_{a_n-\epsilon}^{a_n+\epsilon}[\Gamma(u+\epsilon)+\Gamma(u-\epsilon)-2\Gamma(u)]^2du$$

$$=\frac{a_n^2}{\epsilon^3}\left[\int_{a_n-\epsilon}^{a_n}[u+\epsilon-a_n]^2du+\int_{a_n}^{a_n+\epsilon}[\epsilon-u+a_n]^2du\right]$$

$$=\frac{2a_n^2}{\epsilon^3}\int_0^{\epsilon}u^2du=\frac{2a_n^2}{3}.$$

We thus see that except for a constant factor,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^3}\int_0^{\infty}\{[\Delta(u+\epsilon)+\Delta(u-\epsilon)-2\Delta(u)]^2$$

$$+[\Gamma(u+\epsilon)+\Gamma(u-\epsilon)-2\Gamma(u)]^2\}du$$

will represent in this case

$$\sum_1^{\infty}(a_n^2+b_n^2)$$

or the sum of the squares of the jumps of $\Gamma'(u)$ and $\Delta'(u)$.

§2. **The Coefficient-Functions of the Trigonometrical Development of the Transforms of a Given Function.**

Now let $\Phi(x)$ be any even function which is summable and of summable square over the entire interval $(-\infty, \infty)$ and which is $0(1/x^2)$ at infinity. Then

$$g(t)=\int_{-\infty}^{\infty}f(\tau)\Phi(t-\tau)d\tau \tag{19}$$

exists. We retain the supposition of the previous section, that

$$\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}[f(t)]^2dt$$

exists. We shall have

$$\int_{-A}^{A}[g(t)]^2dt=\int_{-A}^{A}dt\left[\int_{-\infty}^{\infty}f(\tau)\Phi(t-\tau)d\tau\right]^2$$

$$=\int_{-A}^{A}dt\left[\int_{-\infty}^{\infty}f(t-u)\Phi(u)du\right]^2$$

$$=\int_{-\infty}^{\infty}\Phi(u)du\int_{-\infty}^{\infty}\Phi(v)dv\int_{-A}^{A}f(t-u)f(t-v)dt, \tag{20}$$

inasmuch as we can invert the order of integration, all the integrals concerned being uniformly and absolutely convergent. Hence

$$\int_{-A}^{A}[g(t)]^2dt \leq \int_{-\infty}^{\infty}|\Phi(u)|du\int_{-\infty}^{\infty}|\Phi(v)|\,dv\sqrt{\int_{-A}^{A}[f(t-u)]^2dt\int_{-A}^{A}[f(t-v)]^2dt}$$

$$\leq K\int_{-\infty}^{\infty}|\,\Phi(u)\,|du\int_{-\infty}^{\infty}|\,\Phi(v)\,|(A+w)dv, \tag{21}$$

where w stands for the larger of the quantities, $|u|$ and $|v|$. It follows that

$$\frac{1}{2A}\int_{-A}^{A}[g(t)]^2dt$$

is bounded. This is sufficient to prove the existence of $\gamma(u)$ and $\delta(u)$, which bear the same relation to $g(t)$ as $\Gamma(u)$ and $\Delta(u)$ do to $f(u)$. The methods of our first section also lead us to

$$\frac{1}{\pi}\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}[g(t)]^2\frac{\sin^4ht}{t^4}dt=\lim_{\epsilon\to 0}\frac{1}{\epsilon^3}\int_0^{\infty}\Big\{[\delta(u+\epsilon)+\delta(u-\epsilon)-2\delta(u)]^2$$

$$+[\gamma(u+\epsilon)+\gamma(u-\epsilon)-2\gamma(u)]^2\Big\}du \tag{22}$$

if the former limit exists. We have

$$\gamma(u)=\frac{1}{\pi}\int_{-\infty}^{\infty}g(t)\frac{1-\cos ut}{t^2}dt=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos ut}{t^2}\,dt\int_{-\infty}^{\infty}f(t)\Phi(t-\tau)d\tau$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}f(\tau)d\tau\int_{-\infty}^{\infty}\Phi(t-\tau)\frac{1-\cos ut}{t^2}dt$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}f(\tau)d\tau\int_{-\infty}^{\infty}\Phi(t-\tau)\int_0^u dv\int_0^v\cos wt\,dw$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}f(\tau)d\tau\int_0^u dv\int_0^v\cos w\tau\,dw\int_{-\infty}^{\infty}\Phi(x)\cos wx\,dx$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}f(\tau)d\tau\int_0^u dv\int_0^v\phi(w)\cos w\tau\,dw, \tag{23}$$

where

$$\phi(w)=\int_{-\infty}^{\infty}\Phi(x)\cos wx\,dx. \tag{24}$$

Similarly,

$$\delta(u) = C + \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)d\tau \int_0^u dv \int_0^v \phi(w) \sin w\tau \, dw. \quad (25)$$

We have

$$\begin{aligned}
&\gamma(u+\epsilon)+\gamma(u-\epsilon)-2\gamma(u)\\
&\quad=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)d\tau \left[\int_u^{u+\epsilon} - \int_{u-\epsilon}^{u}\right] dv \int_0^v \phi(w)\cos w\tau \, dw\\
&\quad=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)d\tau \int_0^{\epsilon} d\xi \int_0^{u+\xi} \phi(w)\cos w\tau \, dw\\
&\qquad-\frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)d\tau \int_0^{\epsilon} d\xi \int_0^{u+\xi-\epsilon} \phi(w)\cos w\tau \, dw\\
&\quad=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)d\tau \int_0^{\epsilon} d\xi \int_{-\epsilon}^{0} \phi(u+\xi+\eta)\cos(u+\xi+\eta)\tau \, d\eta\\
&\quad=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)d\tau \int_0^{\epsilon} d\xi \left[\frac{\sin(u+\xi)\tau}{\tau}\phi(u+\xi)\right.\\
&\qquad\qquad-\frac{\sin(u+\xi-\epsilon)\tau}{\tau}\phi(u+\xi-\epsilon)\\
&\qquad-\left.\int_{-\epsilon}^{0}\frac{\sin(u+\xi+\eta)\tau}{\tau}\phi'(u+\xi+\eta)d\eta\right]\\
&\quad=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)d\tau \int_0^{\epsilon} d\xi \left\{\left[\frac{\sin(u+\xi)\tau}{\tau}-\frac{\sin(u+\xi-\epsilon)\tau}{\tau}\right]\phi(u+\xi)\right.\\
&\qquad+\frac{\sin(u+\xi-\epsilon)\tau}{\tau}\left(\phi(u+\xi)-\phi(u+\xi-\epsilon)\right)\\
&\qquad+\left[\frac{\cos(u+\xi+\eta)\tau}{\tau^2}\phi'(u+\xi+\eta)\right]_{-\epsilon}^{0}\\
&\qquad-\left.\int_{-\epsilon}^{0}\frac{\cos(u+\xi+\eta)\tau}{\tau^2}\phi''(u+\xi+\eta)\,d\eta\right\}
\end{aligned}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)d\tau \left\{ \left[-\frac{\cos(u+\epsilon)\tau}{\tau^2} + \frac{2\cos u\tau}{\tau^2} - \frac{\cos(u-\epsilon)\tau}{\tau^2} \right] \phi(u) \right.$$

$$+ \left[-\frac{\cos(u+\epsilon)\tau}{\tau^2} + \frac{\cos u\tau}{\tau^2} \right] \left[\phi(u+\epsilon) - \phi(u) \right]$$

$$+ \left[-\frac{\cos u\tau}{\tau^2} + \frac{\cos(u-\epsilon)\tau}{\tau^2} \right] \left[\phi(u+\epsilon) - \phi(u) \right]$$

$$+ \frac{\cos(u-\epsilon)\tau}{\tau^2} \left[2\phi(u) - \phi(u-\epsilon) - \phi(u+\epsilon) \right]$$

$$+ \int_0^{\epsilon} \frac{\cos(u+\xi-\epsilon)\tau}{\tau^2} \left[\phi'(u+\xi) - \phi'(u+\xi-\epsilon) \right] d\xi$$

$$+ \int_0^{\epsilon} \frac{\cos(u+\xi)\tau}{\tau^2} \left[\phi'(u+\xi) - \phi'(u+\xi-\epsilon) \right] d\xi$$

$$+ \int_0^{\epsilon} \frac{\cos(u+\xi)\tau - \cos(u+\xi-\epsilon)\tau}{\tau^2} \phi'(u+\xi-\epsilon)d\xi$$

$$\left. - \int_0^{\epsilon} d\xi \int_{-\epsilon}^{0} \frac{\cos(u+\xi+\eta)\tau}{\tau^2} \phi''(u+\xi+\eta)d\eta \right\}. \tag{26}$$

It follows from this and the Schwarz inequality, together with the existence of $\Gamma'(u)$, that

$$\int_A^B \left\{ \gamma(u+\epsilon) + \gamma(u-\epsilon) - 2\gamma(u) \right.$$

$$\left. - \phi(u) \left[\Gamma(u+\epsilon) + \Gamma(u-\epsilon) - 2\Gamma(u) \right] \right\}^2 du = O(\epsilon^4). \tag{27}$$

Since

$$\int_A^B \left[\Gamma(u+\epsilon) + \Gamma(u-\epsilon) - 2\Gamma(u) \right]^2 du = O(\epsilon^3), \tag{28}$$

we may readily conclude that

$$\int_A^B \left[\phi(u) \right]^2 \left[\Gamma(u+\epsilon) + \Gamma(u-\epsilon) - 2\Gamma(u) \right]^2 du$$

$$- \int_A^B \left[\gamma(u+\epsilon) + \gamma(u-\epsilon) - 2\gamma(u) \right]^2 du = o(\epsilon^3). \tag{29}$$

A similar result holds when we substitute Δ for Γ and δ for γ.

There is a particular class of functions ϕ which is of some importance. Let $\phi_{N\eta}(u)$ be 1 from $u=0$ to $u=N$ and 0 from $u=N+\eta$ onwards. In the intermediate interval from N to $N+\eta$ let $\phi_{N\eta}(u)$ assume values ranging from 1 to 0 in such a manner that $\phi_{N\eta}(u)$ will everywhere possess continuous first, second and third derivatives. It is easy to see that if u, $u-\epsilon$, and $u+\epsilon$ are all greater than $N+\eta$, and if $\phi(u)=\phi_{N\eta}(u)$

$$\gamma(u+\epsilon)+\gamma(u-\epsilon)-2\gamma(u)$$
$$=\frac{2}{\pi}\int_{-\infty}^{\infty}f(\tau)d\tau\int_0^{\epsilon}d\xi\int_{-\epsilon}^{0}\cos(u+\xi+\xi)\tau\ \phi_{N\eta}(u+\xi+\xi)d\xi=0. \quad (30)$$

Similarly,
$$\delta(u+\epsilon)+\delta(u-\epsilon)-2\delta(u)=0. \quad (31)$$

Accordingly, if

$$\frac{3}{2\pi}\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}[g(t)]^2\frac{\sin^4 ht}{t^4}dt$$

exists, and in particular, if

$$\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}[g(\xi)]^2d\xi$$

exists, we have

$$\frac{3}{2\pi}\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}[g(t)]^2\frac{\sin^4 ht}{t^4}\,dt$$
$$=\frac{3}{2}\lim_{\epsilon\to 0}\frac{1}{\epsilon^3}\int_0^{\infty}[\phi(u)]^2\{[\Delta(u+\epsilon)+\Delta(u-\epsilon)-2\Delta(u)]^2$$
$$+[\Gamma(u+\epsilon)+\Gamma(u-\epsilon)-2\Gamma(u)]^2\}du. \quad (32)$$

Let us write $g_{N\eta}(t)$ for the $g(t)$ corresponding to $\phi_{N\eta}(u)$. From this stage on we make the following assumption: *the limit*

$$\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}[g_{N\eta}(t)]^2\frac{\sin^4 ht}{t^4}\,dt \quad (33)$$

shall exist for an everywhere dense set of N's from 0 to infinity, and for each of a denumerable set of η's containing terms of arbitrarily small size. Manifestly if $N_1>N+\eta$, we shall have

$$\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}[g_{N_1\eta_1}(t)]^2\frac{\sin^4 ht}{t^4}\,dt$$
$$\geq\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}[g_{N\eta}(t)]^2\frac{\sin^4 ht}{t^4}dt. \quad (34)$$

Thus the values of

$$\frac{3}{\pi}\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}[g_{N\eta}(t)]^2\frac{\sin^4 ht}{t^4}dt$$

for $N+\eta<L$ will have a finite upper bound, which we shall call $R\{f;\ L\}$. This will be a non-decreasing function of L, and will hence be continuous except for a denumerable set of finite jumps. We put $R\{f;0\}=0$.

Let $\psi(u)$ be any even function of u with continuous first, second and third derivatives, which, together with its first, second and third derivatives, is summable and of summable square. If $Y>Z>0$, we have

$$\begin{aligned}3\overline{\lim_{\epsilon\to 0}}\frac{1}{\epsilon^3}\int_Y^Z[\psi(u)]^2\{&[\Delta(u+\epsilon)+\Delta(u-\epsilon)-2\Delta(u)]^2\\&+[\Gamma(u+\epsilon)+\Gamma(u-\epsilon)-2\Gamma(u)]^2\}du\\&\leq\max_{Y\leq v\leq Z}[\psi(v)]^2[R\{f;Z+\eta\}-R\{f;\ Y-\eta\}].\end{aligned}\tag{35}$$

Similarly,

$$\begin{aligned}3\underline{\lim_{\epsilon\to 0}}\frac{1}{\epsilon^3}\int_Y^Z[\psi(u)]^2\{&[\Delta(u+\epsilon)+\Delta(u-\epsilon)-2\Delta(u)]^2\\&+[\Gamma(u+\epsilon)+\Gamma(u-\epsilon)-2\Gamma(u)]^2\}du\\&\geq\min_{Y\leq v\leq Z}[\psi(v)]^2[R\{f;Z-\eta\}-R\{f;\ Y+\eta\}]\end{aligned}\tag{36}$$

f Y and Z are poi nts of continuity of R, we have

$$\begin{aligned}3\overline{\lim_{\epsilon\to 0}}\frac{1}{\epsilon^3}\int_Y^Z[\psi(u)]^2\{&[\Delta(u+\epsilon)+\Delta(u-\epsilon)-2\Delta(u)]^2\\&+[\Gamma(u+\epsilon)+\Gamma(u-\epsilon)-2\Gamma(u)]^2\}du\\&\leq\max[\psi(v)]^2[R\{f;Z\}-R\{f;\ Y\}]\end{aligned}\tag{37}$$

and

$$\begin{aligned}3\underline{\lim_{\epsilon\to 0}}\frac{1}{\epsilon^3}\int_Y^Z[\psi(u)]^2\{&[\Delta(u+\epsilon)+\Delta(u-\epsilon)-2\Delta(u)]^2\\&+[\Gamma(u+\epsilon)+\Gamma(u-\epsilon)-2\Gamma(u)]^2\}du.\\&\geq\min[\psi(v)]^2[R\{f;Z\}-R\{f;\ Y\}].\end{aligned}\tag{38}$$

It then results from the uniform continuity of $\psi(u)$ that

$$\int_0^\infty [\psi(u)]^2 dR\{f; u\} = 3 \lim_{\epsilon \to 0} \frac{1}{\epsilon^3} \int_0^\infty [\psi(u)]^2 \{ [\Delta(u+\epsilon) + \Delta(u-\epsilon) - 2\Delta(u)]^2 + [\Gamma(u+\epsilon) + \Gamma(u-\epsilon) - 2\Gamma(u)]^2 \} du$$

$$= \frac{3}{\pi} \lim_{h \to 0} \frac{1}{h^3} \int_{-\infty}^\infty \frac{\sin^4 ht}{t^4} \left[\frac{1}{\pi} \int_{-\infty}^\infty f(\tau) d\tau \int_0^\infty \psi(u) \cos u(t-\tau) du \right]^2 dt. \tag{39}$$

Let us consider the functions

$$F_1(t) = f(t+w) + f(t-w), \quad F_2(t) = f(t+w) - f(t-w).$$

Let us put

$$\Psi(t) = \frac{1}{\pi} \int_0^\infty \psi(u) \cos ut \, du;$$

$$G_1(t) = \int_{-\infty}^\infty F_1(t) \Psi(t-\tau) d\tau;$$

$$G_2(t) = \int_{-\infty}^\infty F_2(t) \Psi(t-\tau) d\tau;$$

$$g(t) = \int_{-\infty}^\infty f(t) \Psi(t-\tau) d\tau.$$

We have

$$G_1(t) = \int_{-\infty}^\infty f(t) [\Psi(t-\tau+w) + \Psi(t-\tau-w)] d\tau$$

$$= \int_{-\infty}^\infty f(t) \left\{ \frac{1}{\pi} \int_0^\infty \psi(x) [\cos (t-\tau+w)x + \cos (t-\tau-w)x] dx \right\} dt$$

$$= \int_{-\infty}^\infty f(t) \left[\frac{2}{\pi} \int_0^\infty \psi(x) \cos wx \cos (t-\tau)x \, dx \right] dt. \tag{40}$$

Accordingly,

$$4 \int_0^\infty \cos^2 wu [\psi(u)]^2 dR\{f; u\} = \frac{3}{\pi} \lim_{h \to 0} \frac{1}{h^3} \int_{-\infty}^\infty [G_1(t)]^2 \frac{\sin^4 ht}{t^4} dt.$$

Similarly

$$4 \int_0^\infty \sin^2 wu [\psi(u)]^2 dR\{f; u\} = \frac{3}{\pi} \lim_{h \to 0} \frac{1}{h^3} \int_{-\infty}^\infty [G_2(t)]^2 \frac{\sin^4 ht}{t^4} dt.$$

Hence by subtraction,

$$\int_0^\infty \cos 2wu [\psi(u)]^2 dR\{f; u\} = \frac{3}{\pi} \lim_{h \to 0} \frac{1}{h^3} \int_{-\infty}^\infty g(t+w) g(t-w) \frac{\sin^4 ht}{t^4} dt.$$

If we now let $\psi(u)$ approach 1, while it never exceeds 1 in absolute value,

$$\int_0^\infty \cos 2wu\, dR\{f;u\} = \lim_{\psi(u)\to 1} \frac{3}{\pi} \lim_{h\to 0} \frac{1}{h^3} \int_{-\infty}^{\infty} g(t+w)g(t-w) \frac{\sin^4 ht}{t^4}\, dt. \tag{41}$$

Now,

$$\begin{aligned}
&\frac{1}{h^3}\int_{-\infty}^{\infty} \frac{\sin^4 ht}{t^4}\,[g(t+w)g(t-w)-f(t+w)f(t-w)]\,dt \\
&= \frac{1}{4h^3}\int_{-\infty}^{\infty} \frac{\sin^4 ht}{t^4}\,\{\,[g(t+w)+g(t-w)]^2-[g(t+w)-g(t-w)]^2 \\
&\qquad -[f(t+w)+f(t-w)]^2+[f(t+w)-f(t-w)]^2\,\}\,dt \\
&= \frac{1}{4h^3}\int_{-\infty}^{\infty} \frac{\sin^4 ht}{t^4}\,[g(t+w)+g(t-w)-f(t+w)-f(t-w)] \\
&\qquad [g(t+w)+g(t-w)+f(t+w)+f(t-w)]dt \\
&\quad -\frac{1}{4h^3}\int_{-\infty}^{\infty} \frac{\sin^4 ht}{t^4}\,[g(t+w)-g(t-w)-f(t+w)+f(t-w)] \\
&\qquad [g(t+w)-g(t-w)+f(t+w)-f(t-w)]\,dt \\
&\leq \frac{1}{4h^3}\sqrt{\int_{-\infty}^{\infty} \frac{\sin^4 ht}{t^4}\,[g(t+w)+g(t-w)-f(t+w)-f(t-w)]^2dt} \\
&\qquad \sqrt{\int_{-\infty}^{\infty} \frac{\sin^4 ht}{t^4}\,[g(t+w)+g(t-w)+f(t+w)+f(t-w)]^2dt} \\
&\quad +\frac{1}{4h^3}\sqrt{\int_{-\infty}^{\infty} \frac{\sin^4 ht}{t^4}\,[g(t+w)-g(t-w)-f(t+w)+f(t-w)]^2dt} \\
&\qquad \sqrt{\int_{-\infty}^{\infty} \frac{\sin^4 ht}{t^4}\,[g(t+w)-g(t-w)+f(t+w)-f(t-w)]^2dt} \\
&\leq \frac{2}{h^3}\sqrt{2\int_{-\infty}^{\infty} \frac{\sin^4 ht}{t^4}\,[g(t)-f(t)]^2dt \int_{-\infty}^{\infty} \frac{\sin^4 ht}{t^4}\,[[g(t)]^2+[f(t)]^2]dt} \\
&\leq \frac{4}{h^3}\sqrt{\int_{-\infty}^{\infty} \frac{\sin^4 ht}{t^4}\,[g(t)-f(t)]^2dt \int_{-\infty}^{\infty} \frac{\sin^4 ht}{t^4}\,[f(t)]^2dt} \\
&\leq \frac{4}{h^3}\sqrt{\frac{\pi}{8}\int_0^\infty \{\,[\Delta_1(u+2h)+\Delta_1(u-2h)-2\Delta_1(u)]^2} \\
&\qquad \overline{+[\Gamma_1(u+2h)+\Gamma_1(u-2h)-2\Gamma_1(u)]^2\,\}\,du \int_{-\infty}^{\infty} \frac{\sin^4 ht}{t^4}\,[f(t)]^2dt},
\end{aligned} \tag{42}$$

where

$$\Gamma_1(u) = \Gamma(u) - \gamma(u), \quad \Delta_1(u) = \Delta(u) - \delta(u),$$

and $\psi(u)$ replaces $\phi(u)$ in the definition of $\gamma(u)$ and $\delta(u)$.

Let us give to $\psi(u)$ the particular value $\phi_{N1}(u)$, and let us write $\Phi_{N1}(t)$ for $\Psi(t)$. We have

$$\Gamma_1(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)dt \left\{ \frac{1-\cos ut}{t^2} - \int_0^u dv \int_0^v \phi_{N1}(w)\cos wt\, dw \right\}. \tag{43}$$

If we denote this particular value of Γ_1 by Γ_{1N}, we may readily conclude that

$$\begin{cases} \Gamma_{1N}(u+2h)+\Gamma_{1N}(u-2h)-2\Gamma_{1N}(u)=0 \quad \text{if } u+2h<N; \\ \Gamma_{1N}(u+2h)+\Gamma_{1N}(u-2h)-2\Gamma_{1N}(u)=\Gamma_{1N}(u+2h)+\Gamma_{1N}(u-2h) \\ \quad -2\Gamma_{1N}(u)-\Gamma_{1,N+2}(u+2h)-\Gamma_{1,N+2}(u-2h)+2\Gamma_{1,N+2}(u) \\ \qquad \text{if } u+2h<N+2; \\ \Gamma_{1N}(u+2h)+\Gamma_{1N}(u-2h)-2\Gamma_{1N}(u)=\Gamma(u+2h)+\Gamma(u-2h) \\ \quad -2\Gamma(u) \quad \text{if } u-2h>N+1. \end{cases}$$

With an analogous definition of Δ_{1N}, similar results hold with reference to Δ and its allied functions. As a consequence,

$$\begin{aligned} &\lim_{h\to 0}\frac{\pi}{8h^3}\int_0^\infty \left\{ [\Delta_{1N}(u+2h)+\Delta_{1N}(u-2h)-2\Delta_{1N}(u)]^2 \right. \\ &\qquad \left. +[\Gamma_{1N}(u+2h)+\Gamma_{1N}(u-2h)-2\Gamma_{1N}(u)]^2 \right\} du \\ &\leq \lim_{h\to 0}\frac{\pi}{8h^3}\int_0^\infty \left\{ [\Delta(u+2h)+\Delta(u-2h)-2\Delta(u)]^2 \right. \\ &\qquad \left. +[\Gamma(u+2h)+\Gamma(u-2h)-2\Gamma(u)]^2 \right\} du \\ &\qquad -\frac{\pi}{3}\int_0^\infty [\phi_{1N}(u)]^2 dR\{f;u\} \\ &\qquad +\frac{\pi}{3}\int_0^\infty [\phi_{1,N+2}(u)-\phi_{1N}(u)]^2 dR\{f;u\}. \end{aligned}$$

We see at once that if

$$\lim_{h\to 0}\frac{\pi}{8h^3}\int_0^\infty \{[\Delta(u+2h)+\Delta(u-2h)-2\Delta(u)]^2 + [\Gamma(u+2h)+\Gamma(u-2h)-2\Gamma(u)]^2\}du=\frac{\pi}{3}\int_0^\infty dR\{f;u\},$$

we have

$$\lim_{N\to\infty}\lim_{h\to 0}\int_{-\infty}^{\infty}\frac{\sin^4 ht}{t^4}[g_{1N}(t+w)g_{1N}(t-w)-f(t+w)f(t-w)]dt=0.$$

Under this hypothesis,

$$\int_0^\infty \cos 2wu\, dR\{f;u\}=\frac{3}{\pi}\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}\frac{\sin^4 ht}{t^4}f(t+w)f(t-w)dt. \tag{44}$$

That is, if

$$\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}f(t+w)f(t-w)dt$$

exists, we have

$$\frac{1}{2}\int_0^\infty \cos 2wu\, dR\{f;u\}=\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}f(t+w)f(t-w)dt.$$

Let us now collect together our hypotheses. These are:

(1) The limit

$$\lim_{A\to\infty}\frac{1}{2A}\int_{-\infty}^{\infty}[f(t)]^2dt$$

exists; (2)

$$\lim_{h\to}\frac{1}{h^3}\int_{-\infty}^{\infty}[g_{N\eta}(t)]^2\frac{\sin^4 ht}{t^4}dt$$

exists for an everywhere dense set of values of N, and for a denumerable set of values of η containing arbitrarily small values; (3)

$$\lim_{N\to\infty}\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}[g_{N\eta}(t)]^2\frac{\sin^4 ht}{t^4}dt = \lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}[f(t)]^2\frac{\sin^4 ht}{t^4}dt.$$

It will be seen that these hypotheses involve directly only a denumerable set of transforms of $f(t)$, which is exceedingly important for the application of considerations from the theory of probability. This is the case because the probability that any one of a denumerable set of events of zero probability shall occur is itself zero. The hypotheses of the author's previous paper involve assertions of uniformity which comprise a non-denumerable infinity of discrete cases, and are far more difficult to tie up with probability theory.

Under these hypotheses we have, as we have just shown,

$$\int_0^\infty \cos 2wu \, dR\{f;u\} = \frac{3}{\pi} \lim_{h\to 0} \frac{1}{h^3} \int_{-\infty}^{\infty} \frac{\sin^4 ht}{t^4} f(t+w)f(t-w)dt,$$

where

$$R\{f, L\} = \limsup_{N+\eta < L} \frac{3}{\pi} \lim_{h\to 0} \frac{1}{h^3} \int_{-\infty}^{\infty} [g_{N\eta}(t)]^2 \frac{\sin^4 ht}{t^4} dt.$$

Now let us consider the general case of an expression

$$\int_0^\infty \cos \alpha u \, dH(u) = K(\alpha),$$

where $H(u)$ is an odd non-decreasing function of u, $H(0)=0$, and where $H(\infty) = \lim_{u\to+\infty} H(u)$ is finite. We have

$$\begin{aligned}
\frac{2}{\pi}\int_0^A K(\alpha) \frac{\sin \alpha v}{\alpha} d\alpha &= \frac{2}{\pi}\int_0^A \frac{\sin \alpha v}{\alpha} d\alpha \int_0^\infty \cos \alpha u \, dH(u) \\
&= \lim_{B\to\infty} \frac{2}{\pi}\int_0^A \frac{\sin \alpha v}{\alpha} d\alpha \left\{ \int_0^B \alpha \sin \alpha u \, H(u)du + H(B)\cos \alpha B \right\} \\
&= \lim_{B\to\infty} \frac{2}{\pi}\int_0^A \sin \alpha v \, d\alpha \int_0^B \sin \alpha u \, H(u)du \\
&= \lim_{B\to\infty} \frac{2}{\pi}\int_0^B H(u)du \int_0^A \sin \alpha v \sin \alpha u \, d\alpha \\
&= \frac{1}{\pi}\int_0^\infty H(u) \left[\frac{\sin A(u-v)}{u-v} - \frac{\sin A(u+v)}{u+v} \right] du \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} H(w+v) \frac{\sin Aw}{w} dw. \qquad (45)
\end{aligned}$$

Hence, since $H(u)$ is of limited total variation, we have by Dirichlet's formula

$$H(v)=\frac{2}{\pi}\int_0^\infty K(\alpha)\frac{\sin \alpha v}{\alpha}\,d\alpha. \tag{46}$$

In particular,

$$R\{f;u\} = \frac{2}{\pi}\int_0^\infty \left\{\frac{3}{\pi}\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}\frac{\sin^4 ht}{t^4} f\left(t+\frac{\alpha}{2}\right)f\left(t-\frac{\alpha}{2}\right)dt\right\}\frac{\sin \alpha u}{\alpha}\,d\alpha. \tag{47}$$

A METHOD OF SOLVING POWER NETWORKS BY MEANS OF CONJUGATE VECTORS

By W. V. Lyon and F. L. Hitchcock

Conjugate vectors are useful in attacking two types of problems that occur in Electrical Engineering. One is the determination of transient currents, and the other is the solution of power networks when the power and power factor of the connected loads are given.

Let x be a complex quantity which may represent voltage, current, volt-amperes, impedance or admittance. Then

$$x = a+jb$$

conjugate of x, $\quad x_c = a-jb$

j conjugate of x, $\quad x_{jc} = -a+jb.$

The following relations are seen to hold:

$$(x+y)_c = x_c + y_c, \quad (xy)_c = y_c y_c,$$
$$(x_c)_c = x, \quad (x^n)_c = (x_c)^n,$$
$$x_c = -x_{jc}, \quad (xy)_{jc} = x_c y_{jc}.$$

Furthermore if E and I are the complex quantities representing the r. m. s. voltage applied to and the r. m. s. current in a circuit, then

$$E_c I = P + jQ = W \text{ [1]}$$

where P is the real or active power and Q, the quadrature or reactive power.

In the determination of the magnitude of transient oscillations, the following method is useful. If I is the complex quantity which represents the transient current, its instantaneous value, i, is given by

$$ji = \frac{1}{2}(I + I_{jc}).$$

The instantaneous value of the electric charge, q, is given by

$$jq = \frac{1}{2}\left(\frac{I}{m} + \left(\frac{I}{m}\right)_{jc}\right),$$

where m is the angular velocity of the current, I.

[1] The Measurement of Power in Polyphase Circuits, Charles Fortescue, A. I. E.E., Feb. 1923.

The elimination of I_{jc} from these equations gives

$$I = j(i - qm_c)\frac{2m}{m - m_c}.$$

The following linear equation is of frequent occurrence:

$$Ax + Bx_c = D, \tag{1}$$

where A, B and D are known complexes and x is the unknown. Write the conjugate equation

$$A_c x_c + B_c x = D_c. \tag{2}$$

Eliminate x_c from (1) and (2) and solve for x

$$x = \frac{DA_c - D_c B}{AA_c - BB_c}. \tag{3}$$

Equation (1) is not sufficient if A and B are conjugate, and (3) does not hold.

The problem of a single load of known power and power-factor at the end of a high tension line results in a quadratic equation. The voltage, E, at the load is given by

$$V - IZ = E \tag{4}$$

where V is the voltage at the generating station, I, the line current, and Z, the line impedance. The vector volt-amperes, W, at the load is given by:

$$E_c(I - Ey) = W \tag{5}$$

where y is the line admittance at the load on the assumption of an equivalent π line.

Eliminating I between equations (4) and (5) gives

$$V - E(1 + yZ) = \frac{WZ}{E_c}. \tag{6}$$

Solve equation (6) for E in terms of the knowns and E_c and, after conjugating the result, substitute in (6) for E_c. This gives:

$$V - E(1 + yZ) = \frac{WZ(1 + y_c Z_c)}{V - \dfrac{W_c Z_c}{E}}$$

V can be chosen along the axis of reals so that $V = V_c$.

Expand and collect terms in E,

$$V(1 + yZ)E^2 + (WZ(1 + y_c Z_c) - W_c Z_c(1 + yZ) - V^2)E + VW_c Z_c = 0.$$

The solution for E can be obtained by the usual quadratic formula.

Another interesting problem is that of two loads fed from a single generating station over independent lines but with a tie line between them. In this case we will assume that the line voltage is so low that the line capacitance may be neglected. This, however, does not simplify the solution of the problem in any marked degree. The resulting delta network may be replaced by its equivalent Y, the constants of which are z, z_1, and z_2. The voltage at the generating station is V, the load voltages are E_1 and E_2, the vector voltamperes at the loads are W_1 and W_2, and the load currents are I_1 and I_2. The four known relations then are

$$V-(I_1+I_2)z-I_1z_1=E_1 \tag{7}$$

$$V-(I_1+I_2)z-I_2z_2=E_2 \tag{8}$$

$$E_{1c}I_1=W_1 \tag{9}$$

$$E_{2c}I_2=W_2. \tag{10}$$

The four unknowns being I_1, I_2, E_1, and E_2, these equations may be solved in any given case by the following method. First substitute the values of I_1 and I_2 obtained from (9) and (10) in (7), and clear of fractions, giving

$$VE_{1c}E_{2c}-(W_1E_{2c}+W_2E_{1c})z-W_1E_{2c}z_1=E_1E_{1c}E_{2c}. \tag{11}$$

The product E_1E_{1c} is a real quantity, which we may denote by x. By solving (11) for E_{2c} we have

$$E_{2c}=\frac{W_2E_{1c}z}{VE_{1c}-a_1W_1-x} \tag{12}$$

where for brevity the known quantity $z+z_1$ has been denoted by a_1.

Equations (7) and (8) may be written in the form

$$a_1I_1+zI_2=V-E_1 \tag{13}$$

$$zI_1+a_2I_2=V-E_2 \tag{14}$$

where $a_2=z+z_2$. Letting $a_1a_2-z^2=p$, the value of I_1 from these equations is given by

$$pI_1=z_2V-a_2E_1+zE_2. \tag{15}$$

Solving for E_2 and taking conjugates,

$$z_c E_{2c} = p_c \frac{W_{1c}}{E_1} - z_{2c} V + a_{2c} E_{1c}. \tag{16}$$

We now eliminate E_{2c} between (12) and (16), giving easily

$$W_2 z z_c x = (V E_{1c} - a_1 W_1 - x)(p_c W_{1c} - z_{2c} V E_1 + a_{2c} x), \tag{17}$$

a reasonably simple equation which is free from the two currents I_1 and I_2 and from one of the voltages. Since $x = E_1 E_{1c}$ we might regard this as a cubic equation in E_1 and E_{1c} together, and by taking conjugates obtain simultaneous equations to solve for E_1 and E_{1c}. It is much better, however, to proceed as follows: By collecting terms in E_1 and E_{1c}, writing $x = E_1 E_{1c}$ wherever this product occurs we obtain

$$E_1 z_{2c} V(a_1 W_1 + x) + E_{1c} V(p_c W_{1c} + a_{2c} x) = a_{2c} x^2 + (p_c W_{1c} + z_{2c} V^2 + a_1 a_{2c} W_1 + W_2 z z_c) x + a_1 p_c W_1 W_{1c} = D, \text{ say} \tag{18}$$

so that if we call the coefficients of E_1 and E_{1c} respectively A and B, the equation takes the form

$$A E_1 + B E_{1c} = D \tag{19}$$

where A and B are known linear functions of x and D is a known quadratic in x. Solving for E_1 as in (3) we have

$$E_1 = \frac{D A_c - D_c B}{A A_c - B B_c} \tag{20}$$

where the numerator is a known cubic in x and the denominator is a known quadratic in x. By taking conjugates,

$$E_{1c} = \frac{D_c A - D B_c}{A A_c - B B_c} \tag{21}$$

and by multiplying (21) into (20) we have

$$x = \frac{D D_c A A_c - (D^2 A_c B_c + D_c^2 A B) + D D_c B B_c}{(A A_c - B B_c)^2} \tag{22}$$

where both numerator and denominator are real. The numerator is of the sixth degree in x and the denominator is of the fourth degree. Thus we have a real equation of the sixth degree in x which can be solved by Horner's method. The value of E_1 is then given by (20), that of E_2 by (12), and finally I_1 and I_2 are found by (9) and (10). The problem is thus solved.

NOTE ON A METHOD OF EVALUATING THE COMPLEX ROOTS OF A QUARTIC EQUATION

By Y. H. Ku

There was recently published a note on a method of solving a quartic equation which has two pairs of conjugate roots.[1] The following more general solution resulted from a further study of the problem, particularly of the case in which the coefficients of the quartic are complex numbers. The basis of the present solution is the determination of the two quadratic factors of the quartic. Let the quartic be:

$$a+bx+cx^2+dx^3+x^4=0. \tag{1}$$

Let its two quadratic factors be:

$$(A+Bx+x^2)(C+Dx+x^2)=0. \tag{2}$$

If m represents the sum of A and C, then it can be shown that m is one of the roots of the cubic:

$$4ac-ad^2-b^2+(bd-4a)m-cm^2+m^3=0.^2 \tag{3}$$

This is a general reducing cubic of the quartic which holds whether the coefficients are real or complex.

Case I. Real Coefficients.

When the coefficients of the quartic are real, the coefficients of the reducing cubic are also real and its solution is best determined by some successive approximations method.

When a is positive, one of the roots is real and somewhat greater than $2\sqrt{a}$. Let m be this root, then,

$$A \text{ or } C=\frac{m\pm\sqrt{m^2-4a}}{2} \tag{4}$$

and

$$B \text{ or } D=\frac{d\pm\sqrt{d^2-4c+4m}}{2}. \tag{5}$$

[1] Note on a Method of Evaluating the Complex Roots of a Quartic Equation, W. V. Lyon, Journal of Mathematics and Physics, Vol. III, No. 3, April 1924.

[2] This the same cubic that appears in Lyon's note, although it is derived in an entirely different way.

The proper signs to be used in determining A, B, C and D must be such as will make

$$BC+AD=b. \tag{6}$$

The roots of the quartic are now given by the usual quadratic formula

$$x=\frac{-B\pm\sqrt{B^2-4A}}{2} \tag{7}$$

and

$$x=\frac{-D\pm\sqrt{D^2-4C}}{2} \tag{8}$$

Case II. Complex Coefficients.

In this case, the coefficients of the reducing cubic are complex, and the solution can be obtained by Cardan's method. If, however, there are not more than two unrelated complex roots of the quartic, one of the quadratic factors will have real coefficients and the solution is much more easily obtained by the following device:

Let the two quadratic components now be

$$(A+Bx+x^2)(C_1+jC_2+(D_1+jD_2)x+x^2)=0.$$

If the coefficients of the quartic are written in the form

$$a_1+ja_2$$

then it can be shown that there is a characteristic cubic for m similar to equation (3), except that the coefficients are combinations of the real components of the complex coefficients of the quartic. In this case, however, $m=A+C_1$.

$$4a_1c_1-a_1d_1^2-b_1^2+(b_1d_1-4a_1)m-c_1m^2+m^3=0. \tag{9}$$

Having determined a real root, m, of equation (9), the coefficients of the quadratic components are given by

$$A \text{ or } C_1=\frac{m\pm\sqrt{m^2-4a_1}}{2}$$

$$B \text{ or } D_1=\frac{d_1\pm\sqrt{d_1^2-4c_1+4m}}{2}.$$

The proper signs to be used in determining A, C_1, B and D_1 must be such as will make $AD_1+BC_1=b_1$. The imaginary components of C and D are

$$C_2=\frac{a_2}{B}$$

$$D_2=d_2.$$

At this point it is necessary to determine whether the quartic is one for which this method of solution is valid. The test is simple. It is that $\frac{b_2}{B}-\frac{A}{B}d_2=c_2-Bd_2=\frac{a_2}{A}$ or the method fails. In fact, these two equations may be used to determine A and B instead of equation (9). If B is eliminated, the expression for A is

$$A^3-\frac{b_2}{d_2}A^2+\frac{a_2c_2}{d_2^2}A-\frac{a_2^2}{d_2^2}=0.$$

Having determined the two quadratic factors, the roots of the quartic are obtained by the usual quadratic formula.

THE FORCE BETWEEN MOVING CHARGES

By V. Bush

Summary. A theory of the force between moving charges, originally treated by Gauss and Weber, is developed. It involves a variant charge and an invariant mass, and is hence contrary to restricted relativity. It leads correctly to all our laws for electric circuits without the agency of a magnetic field. Some of the immediate objections to this scheme of attack are treated. It is considered in the light of the Kaufmann-Bucherer results. Applied to atomistics it gives the same results as were obtained by Sommerfeld for fine structure by the use of restricted relativity. It gives, for an electron revolving about a nucleus, a critical diameter within which the electron may become imprisoned, and this diameter is computed to be of the order of magnitude believed experimentally to be the diameter of the atomic nucleus.

1. In the following paper I will present a line of attack upon electrical and atomistic phenomena, which in its fundamental elements is as old as classic electromagnetic theory itself,[1] but which has been of late much neglected although it offers an alternative line of approach which may possibly bring together some matters now widely divergent in our theories.

The essential features of this alternative treatment are the abandonment of the concept of the magnetic field, and the use of an invariant mass.

The force between moving charges will be expressed in terms of one component only, and that falling along the line joining the charges.

We will first review a few matters in regard to classic theory. The alternative attack will then be presented, and brought to bear somewhat on electron theory and atomistics, which were unknown at the time this alternative line of thought was first started and then largely forgotten.

[1] Maxwell, Elec. and Mag. Part IV, Chap. II.

Classic electromagnetic theory is based on the implicit assumption of the invariance of charge. In building up the now classic relations between circuits, and the Ampere and Faraday laws, the electrostatic force lying along the line joining the charges was assumed to be independent of their relative motion. Any additional force due to their relative motion was considered to be produced by motion in a magnetic field and to be perpendicular to the velocity and to the field. This is involved in the Biot-Savart law for the force between elements of circuits carrying current. When applied to two individual charges in relative motion, it results in forces which are not in the same line, and which hence have an unbalanced couple, so that Newton's relations do not hold.

In this early formulation the total force between moving charges was split into two parts, attributed respectively to electrostatic and electromagnetic fields, and the former was assumed to be dependent only on the amount of the charges and their separation.

The Biot-Savart law for the magnetic field due to a current element, together with the law for the force on an element of current-carrying circuit located in a magnetic field, are sufficient, but not necessary, as will be considered below, to account for all the observed facts in regard to the forces between compiete circuits carrying current. If we also assume conservation of energy, we can then arrive at the usual expression for the voltage induced in an element of conductor which moves in a magnetic field, and from this deduce all observed facts in regard to induced voltages in systems in which all currents are in closed circuits and of constant value.

All this is involved with the invariant charge assumption, and corresponding to this treatment there is a definite rule for the force between individual moving charges, which is our usual classic rule.

Again applying conservation of energy, we can obtain the law for the voltage induced in a circuit element situated in a varying magnetic field, and hence arrive at induced voltages due to varying currents. When we deal with varying currents, we treat of accelerated electrons. Hence, this last step involves putting in a term in the expression for the force between individual charges

which contains their relative accelerations, although in the classic treatment it is not always realized that this is what we are doing.

Historically, it is quite evident why classic theory was built up in its present form, for electrostatic and magnetic effects were independently known before their inter-relationships were examined, and were considered entirely separate entities. The Ampere and Faraday laws were experimental laws. The rules for circuit elements which would lead to these, and which considered charge invariant, were the natural ones to use. Even although other laws for elements, and hence for individual charges, were possible, they received comparatively little attention, were abandoned entirely upon Maxwell's success with the electromagnetic theory of light, and have not been reëxamined to my knowledge since the time when it appeared that this theory needed review.

Maxwell used the relations between currents and magnetic fields in which the force on a circuit element due to the fact that it carried current and was situated in a magnetic field was taken perpendicular to the element. This is based, when interpreted in terms of individual charges, on the assumption of invariance of charge, that is the electrostatic forces are considered independent of velocity. To this he added his famous assumption regarding displacement currents, and derived his light equations.

Our whole classic electromagnetic theory is thus based on invariance of charge.

In contradiction to this formulation we will proceed on an alternative assumption in which the perpendicular component is absent, and in which consequently the entire burden of accounting for the observed effects falls upon a variant force along the line joining the charges. We thus abandon the magnetic field, and introduce a variant charge, or better, a variant force of electrostatic attraction in its stead.

Just as in relativity we deal with longitudinal and transverse mass, so in this formulation we deal with longitudinal and transverse departures of the electrostatic attraction from that given by Coulomb's law.

2. There are other assumptions than the classic for the force between elements, and hence for the force between charges, which

will lead to all our observed facts in regard to the forces and induced voltages in complete circuits. This is shown by the work of Ampere, Gauss, Weber, Newmann, Riemann, Ritz, and others. This work has been largely forgotten on account of the great success, until recently, of what we here call the classic treatment.

This present paper is not occasioned by any observed or fancied departure of the behavior of complete circuits from that given by the classic equations of the electric and magnetic circuits, for the alternative attack here treated will give to a first order of approximation exactly those same results for complete circuits or even for one complete circuit and an element of another. It is true that in second order effects there will be differences, but I have not yet succeeded in finding such an effect of observable magnitude by which the two theories might be compared. For all measureable effects in complete circuits they are identical in results.

Briefly, the reason different laws for circuit elements are possible is this: Suppose we have an expression for the force between circuit elements which will lead correctly to the observed force between complete circuits. To this expression suppose we add a term which will integrate to zero about any closed path. That is, suppose we add any potential function to the expression for vector potential. The modified expression will then give, not only the same result for two complete circuits, but it will also give the same result for one complete circuit and an element of the second.

We may divide electrical experiments into two classes: (*a*) those on complete circuits, or on a complete circuit and an element of a second, under which is included the effect of a complete circuit on a single moving charge; (*b*) those involving the mutual effects of individual charges. Class (*a*) experiments are incapable of distinguishing between postulates regarding individual charges, provided they lead to the same laws for the force between complete circuits. An important note is this: under class (*a*) we must include the Kaufmann-Bucherer experiment. Below I will show this to be in accord, at least to second power terms, with the present assumption, as well as the classic one of invariant charge, by detailed computations of forces and deflections. Such an accord, however, appears to follow simply from the above discussion alone.

As class (*b*) experiments, we have the vast fund of experimental

evidence regarding atomic structure. This clashes violently with classic electromagnetic theory. That it is in accord with some other assumption than the one at the basis of classic theory appears a foregone conclusion, since it must be in accord with something. The development of the present parallel theory to the point where it can be thus tested has not as yet been made in any completeness. It involves the repetition on a different basis of a vast amount of work. It may even involve several such pieces of analysis, since various alternative fundamental assumptions are possible. I have taken one of these a little way with promising results, and will attempt to show briefly why such work is necessary, and indicate something of what it may produce.

3. If we abandon invariance of charge, do we not abandon restricted relativity? We surely will depart from its present form, but this is a departure the necessity for which is becoming quite apparent, especially in view of Miller's work on Mt. Wilson.

The interconnection between electromagnetic theory and restricted relativity is intimate, but they are not equivalent. The classic electromagnetic equations are invariant under the Lorentz transformation, but it does not follow that the use of these equations makes restricted relativity inevitable. Also, even although we abandon these equations, it does not necessarily follow that we abandon relativity. Still it appears from what follows that we will surely at least profoundly modify it. Such a modification would be involved in placing the burden of variation with velocity fundamentally upon charge rather than mass. In this present formulation all forces will be expressed in terms of relative velocities and accelerations. Thus we deal with a relationship with relative motion only involved, and in which a variant charge replaces a variant mass.

Popularly the Kaufmann-Bucherer results have been interpreted as a substantiation both of restricted relativity and of classic electromagnetic theory. They will, however, as we shall see, lend just as ready support to a formulation in which the reason for the variation of e/m with velocity is assigned to the charge rather than to the mass. Of course classic electromagnetic theory leads to the observed variation of this ratio only when very definite assumptions are made as to the nature of the electron.

Have we, then, no experimental basis for discrimination between restricted relativity as it is and as it becomes if we start with a variant charge, or between classic electromagnetic theory and an alternative development in which the forces between individual charges due to relative uniform motion are equal and opposite and along the line joining them? The vast amount of spectroscopic and similar information concerning the interaction of charges in the atom furnishes just this basis, and it is here that the comparison must finally be made.

It has always been assumed, at least popularly, that the somewhat bizarre results of relativity could be tested experimentally only by the use of very long distances, as in astronomy, or the use of very high velocities, as those of the beta particles in the Kaufmann-Bucherer results.

Really the test must come on the interactions observed between individual charges. Still it is noteworthy that, when we formulate expressions for the action between circuits in terms of the forces between individual charges, we are actually accounting for the observed facts by reason of departures from Coulomb's law due to velocities, and that these departures are of the same order as those given by relativity relationships for variation of magnitudes with velocity. Thus we actually account for the enormous forces brought into play in a dynamo electric machine in terms of small departures from an inverse square law. If the velocity of drift of the electrons of the current is about one centimeter per second, and we deal with corrective terms of the magnitude of those of relativity, the departure from Coulomb's law is given by a factor of about $(1+10^{-21})$. Still, on account of the fact that the first term is very exactly canceled in all experiments with neutral conductors by the presence of an equal number of positive charges, such a factor is nevertheless adequate irrespective of whether we follow classic theory, the present alternative, or some other.

Thus our circuit laws are expressions for the mass effect of streams of charges to be obtained from the law for the force between individual charges. We cannot change that law for individual charges, even in terms involving the square of the velocity, without profoundly affecting the results we will obtain for complete circuits. This is not always clearly realized.

4. Ampere derived an expression for the force between circuit elements which he showed was necessary and sufficient to account for observed facts in regard to forces between complete circuits, under the limitation that the force between elements was assumed to lie entirely in the line joining them. This expression is

$$f = \frac{ii'dldl'}{r^2}(2\cos\theta\cos\theta'\cos\gamma - \sin\theta\sin\theta')$$

where θ and θ' are the angles made by the elements with the perpendiculars to the line joining them, and γ is the angle between planes of the elements. The currents are in electromagnetic units.

I will attempt first to get at the simplest law for the force between charges which will produce this result. I hence assume it to be an inverse square law, the correction terms to be added to the usual electrostatic law being dependent only upon the components of the relative velocity of the charges. This gives an expression of the form

$$\frac{ee'}{r^2}(1 + Av^2\cos^2\theta + Bv^2\sin^2\theta)$$

where e and e' are the charges, r is the separation, and θ is the angle between their relative velocity v and the perpendicular to the line joining them. A and B are as yet undetermined coefficients. This is the simplest symmetrical inverse square law we can write, assuming at present that terms involving powers of the velocity components greater than the square may be neglected.

Assume two elements of circuits dl and dl' with n and n' free electrons per unit of length moving at velocities v and v'. The distance between the elements is r, and they make angles θ and θ' with the perpendicular to the line joining them. The angle between the planes of the angles θ and θ' is γ.

Now the force between these elements in terms of our above expression for the force between charges is the sum of three forces: between the two sets of moving electrons; between the moving electrons of dl and the positive charges, not counterbalanced by stationary electrons, of dl'; and vice versa.

For the third we have in accordance with our expression

$$f_3 = \frac{e^2nn'dldl'}{r^2}(Av'^2\cos^2\theta' + Bv'^2\sin^2\theta')$$

leaving out the Coulomb term which we know to cancel the effect of positive charges, and for the second

$$f_2 = \frac{e^2 nn' dld l'}{r^2} (Av^2 \cos^2 \theta + Bv^2 \sin^2 \theta).$$

For the force between the moving electrons we have to consider a relative velocity:

$$v_0^2 = v^2 + v'^2 - 2vv' (\sin \theta \sin \theta' + \cos \theta \cos \theta' \cos \gamma)$$

and the angle of the relative velocity with the line joining elements is given by

$$\tan \theta_0 = \frac{v' \sin \theta' - v \sin \theta}{\sqrt{v'^2 \cos^2 \theta' - 2vv' \cos \theta \cos \theta' \cos \gamma + v^2 \cos^2 \theta}}$$

and so

$$f_1 = -\frac{e^2 nn' dld l'}{r^2} (Av_0^2 \cos^2 \theta_0 + Bv_0^2 \sin^2 \theta_0)$$

$$= -\frac{e^2 nn' dld l'}{r^2} (Av'^2 \cos^2 \theta' - 2Avv' \cos \theta \cos \theta' \cos \gamma$$

$$+ Av^2 \cos^2 \theta + Bv'^2 \sin^2 \theta' + Bv^2 \sin^2 \theta - 2Bvv' \sin \theta \sin \theta').$$

For the total force between elements

$$f = f_1 + f_2 + f_3 = \frac{e^2 nn' dld l'}{r^2} (2Avv' \cos \theta \cos \theta' \cos \gamma + 2Bvv' \sin \theta \sin \theta')$$

Now if i and i' are the currents in the two elements

$$i = n e v$$

$$i' = n e v'$$

in electrostatic units. If the currents are in electromagnetic units we must divide by our experimentally determined conversion factor c, which here means nothing else, giving

$$i = n e \frac{v}{c}$$

$$i' = n e \frac{v'}{c}$$

or abbreviating by

$$\beta = \frac{v}{c}$$

$$\beta' = \frac{v'}{c}$$

we have

$$i = n e \beta$$

$$i' = n e \beta'$$

Our expression for the force between elements is then

$$f = \frac{ii' dl dl'}{r^2} (2Bc^2 \sin\theta \sin\theta' + 2Ac^2 \cos\theta \cos\theta' \cos\gamma)$$

Compare this with the Ampere expression above, and we see that our undetermined coefficients must have the values

$$A = \frac{1}{c^2}$$

$$B = -\frac{1}{2c^2}$$

Thus we arrive at

$$\frac{ee'}{r^2} (1 + \beta^2 \cos^2\theta - \frac{1}{2} \beta^2 \sin^2\theta)$$

as the simplest expression for the force between moving charges, symmetrical, inverse square, in terms of components of relative velocity only, with the force lying entirely along the line joining the charges, which will lead to the Ampere relation and hence to the usual laws for the forces and induced voltages in complete circuits carrying steady currents.

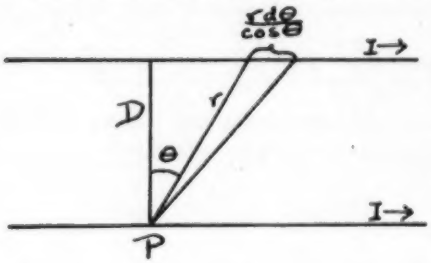
The expression contains as yet no term involving accelerations, and we know it must have such terms in order to account for induced voltages due to varying currents. We do not need this term when currents are steady.

In the development of classic theory such a term was implicitly introduced as soon as laws for induced voltages due to varying currents were considered, although it is not usually written in

this connection. The accelerational term for our expression will be treated below.

5. It is worth while to apply our expression to a special example, for we have become so wedded to the magnetic field that it is important to realize that we may derive our usual results for the forces between circuits without using it at all. Also, the fact that corrections of the order of β^2 can lead to forces which are correct and in accord with experience on circuits is worth emphasis.

Let us compute the force acting between two long parallel wires each carrying a current I. We will compute the vertical force per centimeter length at P due to n free electrons per centimeter length of wire moving at velocity v along the wire. For the element $rd\theta/\cos\theta$ there are $\frac{n\,rd\theta}{\cos\theta}$ free electrons, and due to these an excess force of attraction on the n positive charges per centimeter at P



$$\frac{n^2e^2}{r^2}\beta^2\frac{rd\theta}{\cos\theta}(\cos^2\theta-\frac{1}{2}\sin^2\theta)$$

with a vertical component

$$\frac{n^2e^2\beta^2}{r}(\cos^2\theta-\frac{1}{2}\sin^2\theta)d\theta$$

so that the total force on the centimeter length at P is

$$F=\int_{-\pi/2}^{\pi/2}\left(\frac{2n^2e^2\beta^2}{D}\cos^3\theta\,d\theta-\frac{n^2e^2\beta^2}{D}\sin^2\theta\cos\theta\,d\theta\right)$$

$$=\frac{2\,n^2e^2\beta^2}{D}\frac{\sin\theta}{3}(\cos^2\theta+2)-\frac{n^2e^2\beta^2}{D}\frac{\sin^3\theta}{3}\Bigg]_{-\pi/2}^{\pi/2}$$

$$=2\frac{n^2e^2\beta^2}{D}$$

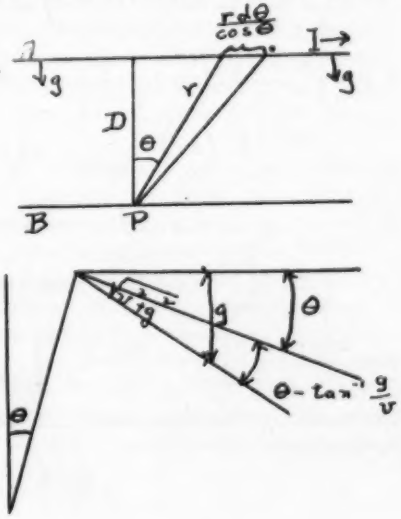
But $n\,e\,\beta=I$ in the electromagnetic system.

So $F = \frac{2I^2}{D}$ dynes per centimeter length, which is correct.

Note that, not only may we arrive at the correct force for two complete circuits, but also for the correct force between one circuit and an element of the other.

6. In classic electromagnetic theory the induced voltage, when circuits move and currents are steady, is given readily for the induced voltage in a complete circuit. Confusion arrives, however, when consideration is given to the voltage induced in a circuit which is open. The problem of when a magnetic field moves, and when it does not, renders analysis unsatisfactory. This difficulty does not arise in the present formulation.

If we still treat a case where currents are constant, our rule above should be sufficient in itself, for no accelerations are involved. Where currents are constant the only electron accelerations are those due to heat motion, the effects of which should cancel out; and accelerations due to actual curvature of the circuit itself, which are neglected both in classic treatment and in the present formulation, and the effect of which is apparently undetectable. Hence, our expression should be sufficient to treat a case where a voltage is induced in a uniformly moving wire, all currents being steady.



Let us compute the voltage induced when a long straight conductor is parallel to a moving second conductor which carries current. For simplicity, we will consider the current-carrying

conductor to move, as it makes no difference which we choose.

We now have to compound velocities.

We consider the force acting on an electrostatic unit of charge at P.

There are $nr\,d\theta/\cos\theta$ free electrons in the element under consideration, and an equal number of positive charges which are fixed with respect to A, but move at velocity g with respect to B.

The excess force due to these is:

$$\frac{ner\,d\theta}{\cos\theta}\frac{(v^2+g^2)}{c^2r^2}\cos^2\left(\theta-\tan^{-1}\frac{g}{v}\right)-\frac{ner\,d\theta\,g^2}{\cos\theta\,c^2r^2}\sin^2\theta$$

$$-\frac{1}{2}\frac{ner\,d\theta}{\cos\theta}\frac{(v^2+g^2)}{c^2r^2}\sin^2\left(\theta-\tan^{-1}\frac{g}{v}\right)+\frac{1}{2}\frac{ner\,d\theta\,g^2}{\cos\theta\,c^2r^2}\cos^2\theta.$$

We wish the component along the wire B, so we will multiply by $\sin\theta$ and integrate

$$\int_{-\pi/2}^{\pi/2}\left[\frac{ne(v^2+g^2)}{c^2D}\cos^2\left(\theta-\tan^{-1}\frac{g}{v}\right)\sin\theta\,d\theta-\frac{eg^2n}{Dc^2}\sin^3\theta\,d\theta\right)$$

$$-\frac{1}{2}\frac{ne(v^2+g^2)}{c^2D}\sin^2\left(\theta-\tan^{-1}\frac{g}{v}\right)\sin\theta\,d\theta+\frac{eg^2n}{2Dc^2}\cos^2\theta\sin\theta\,d\theta\Bigg]$$

$$=\int_{-\pi/2}^{\pi/2}\left[\frac{ne(v^2+g^2)}{c^2D}\left(\sin\theta\frac{g}{\sqrt{v^2+g^2}}+\cos\theta\frac{v}{\sqrt{v^2+g^2}}\right)^2\sin\theta\,d\theta\right.$$

$$\left.-\frac{ne(v^2+g^2)}{2c^2D}\left(\sin\theta\frac{v}{\sqrt{v^2+g^2}}-\cos\theta\frac{g}{\sqrt{v^2+g^2}}\right)^2\sin\theta\,d\theta\right]$$

$$=\frac{ne}{c^2D}\int_{-\pi/2}^{\pi/2}(g^2\sin^3\theta\,d\theta+2vg\sin^2\theta\cos\theta\,d\theta+v^2\cos^2\theta\sin\theta\,d\theta)$$

$$-\frac{ne}{2c^2D}\int_{-\pi/2}^{\pi/2}(v^2\sin^3\theta\,d\theta-2vg\sin^2\theta\cos\theta\,d\theta+g^2\cos^2\theta\sin\theta\,d\theta)$$

$$=+\frac{2vgen}{c^2D}\frac{\sin^3\theta}{3}\Bigg]_{-\pi/2}^{\pi/2}+\frac{vgen}{c^2D}\frac{\sin^3\theta}{3}\Bigg]_{-\pi/2}^{\pi/2}=+\frac{2vgen}{c^2D}.$$

The current in electromagnetic units is $I=ne\beta$. For the force we then have:

$$+\frac{2I}{D}\frac{g}{c}.$$

This force is in dynes per electrostatic unit of charge, that is in statvolts per centimeter. To obtain in abvolts per centimeter, we multiply by c. The induced voltage then is

$$+\frac{2I}{D}g \text{ abvolts per centimeter length}$$

which is in accord with experiment and in the correct direction. The direction may readily be checked by qualitative reasoning.

7. Now, having proceeded thus far, I discovered that an expression given by Gauss,[2] in a memorandum published only after his death, is the same as the expression for the force between charges which we have used above, although he did little with it. Undoubtedly he stopped because the expression is first incomplete; and second, in its present form, inconsistent with conservation of energy.

This matter now appears in a much different light than it did at the time of Gauss. In dealing with the question of conservation of energy we would need to consider accelerational terms and radiation. In the absence of such terms it would be possible to so move a charge in the neighborhood of other charges as to continuously derive energy, but in executing such a motion radiation would be caused and corresponding additional forces introduced. We shall also see below that, even in the absence of such terms, the free motion of an electron about a nucleus would be conservative.

It will be well, though, to consider here one or two other objections which were raised against a formulation such as we are here using, principally by Maxwell and Helmholtz, and which can be treated before accelerational terms are taken up.

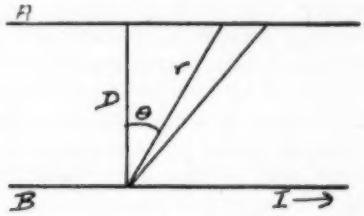
Maxwell,[3] in discussing Ampere's hypothesis, shows that in order that we may have a law for the force between charges which will lead to Ampere's results for the force on circuits, it is necessary either that the current in the wire consist of equal charges moving in opposite directions, or else that the product of the charges by the square of their velocities be the same. He immediately dismisses the first possibility as it would lead to a force

[2] Werke, Gottingen, 1867, vol. 5, p. 616.
[3] Elec. and Mag. Part IV, Chap. XXIII.

between a charged wire not carrying current and a second wire carrying current but not charged, for such a force had not been observed. He therefore pays attention to the second possibility only. But the first arrangement is exactly what we now believe we have in metallic conduction. We deal with relative velocities only, of course, and the effect of any stationary electrons is canceled by the same number of stationary positive charges.

Let us examine, however, the objection of Maxwell and see whether the predicted force is experimentally detectable.

Consider wire A charged to a potential E, and wire B to carry a current I.



We have on A a charge of q electrons per centimeter length.

The force between wires is obtained by the same process as used above, except that we come out with

$$F = \frac{nqe^2\beta^2}{D} \text{ dynes per centimeter length}$$

or since $I = n e \beta$,

$$F = \frac{q e \beta I}{D} \text{ dynes per centimeter length}$$

$Q = q e$ is the electrostatic charge per centimeter

$$F = \frac{Q \beta I}{D}.$$

Now this is peculiar. It contains β, and if the force could be measured we would find it possible to determine β and hence the number of free electrons in the wire.

But let us take v at about 1 cm./sec. for the present, which is

surely as large as its probable value in accord with most of our experimental evidence of its magnitude, so that

$$\beta = \frac{1}{3\times 10^{10}}.$$

Assume

$I = 100$ abamperes

$Q = 10$ electrostatic units per centimeter length

$D = 1$ cm.

Then

$$F = \frac{10^3}{3\times 10^{10}} = \frac{1}{3} \times 10^{-7} \text{ dynes per centimeter length.}$$

It would be very difficult indeed to observe this, for we would have a resistance drop in potential along the current-carrying wire to contend with. Thus there is here no difficulty in the way of our present theory. It is easy to see, though, why Maxwell objected as he did, for although he did not give an estimate of the magnitude of this effect, in his time the velocity of charge along a circuit carrying current was usually thought to be very large.

Another point which seems at first to offer difficulty is the following: According to our theory, all forces between charges are electrostatic in origin. Hence it would seem to be possible to shield one circuit carrying current from forces due to neighboring circuits by surrounding it by an electrostatic shield. Thus, if we have a coil A and a coil B each carrying current, and we enclose B in a metal box, will we not remove or greatly reduce the force between them? Consideration will show this effect also to be negligible. A shield produces its effect only by reason of induced charges on its surface, the net force at a point on its interior, in the absence of any charged bodies inside the shield, being then zero because it is the sum of the effects due to all charges including the induced ones. Now, in our case there will be an induced charge on the shield, but by the reasoning just above it will be negligible in amount. Hence the effect on the interior coil B is as accurately as we can measure the same as though the shield were not present.

It is well also to consider heat motion at this time. If we have a set of equal numbers of positive and negative charges all in chaotic

motion, then evidently if their mean velocities are identical the resulting configuration will also be neutral. But if the negative charges have a higher mean velocity than the positive charges, a unit negative charge exterior to the set will be acted upon by a repulsion when the charges are in motion, for the additive terms in our expression preponderate. The amount of this effect may be obtained by integrating over the unit sphere about a single unit moving charge. This integration gives readily for the total lines of force

$$4\pi\left(1+\frac{1}{2}\beta^2\right)$$

so that the external effect noted above is given by

$$\frac{1}{2}\beta^2 .$$

When a neutral atom is set in motion as a whole, there will evidently be no effect to make it appear non-neutral. When heat motion occurs in a solid body the atoms take part in it as units, as is shown by specific heat experiments. If this were not so, a piece of metal isolated and heated would acquire a very large apparent charge. But only the free electrons take part individually, and we do not know whether even they acquire the velocities dictated by equipartition of energy, nor how many free electrons there are. Hence, we cannot say offhand whether this effect is detectable at all. A rough computation in terms of the usual figures for number of free electrons indicates that it ought to be barely measurable. A series of experiments which have been carried out in an attempt to detect it has not succeeded in finding it positively. Possibly the effect is cancelled by accelerational terms, which have not yet been here introduced.

The host of thermal electric effects, such as the Thomson effect, and so on, may themselves be manifestations of this apparent increase of charge with increase of velocity of the chaotic motion of free electrons. At any rate, this present theory should be of aid in trying to interpret these interactions of heat and electrical effects in conductors, for classic theory does not get far in an attack on them.

The concept of lines of force has little usefulness here as yet. This theory we are developing is no more an "action at a distance" theory than the classic scheme, but the classic system of fields naturally does not describe it. We have set up no mechanism for visualizing the means by which the force we treat is transmitted through a medium. Such a scheme can readily be set up if it makes any one any happier. It appears, however, to be difficult to set up a scheme in terms of fields which will be really useful. This must wait awhile. Still the absence at this point of any interpretation in terms of fields should not be taken to mean that the present formulation is any more involved with action at a distance than is the classic scheme. A development of a wave theory of light along the lines of the present argument may be possible, but I have not attempted it.

8. We will now proceed to examine the Kaufmann-Bucherer results in detail.

An electron moving parallel to a straight wire carrying current is acted upon by a force perpendicular to the direction of motion. If β' gives the velocity ratio for the electron, other quantities being as above, this force is given by

$$F = \int_{-\pi/2}^{\pi/2} \frac{e^2 n r}{r^2 \cos\theta} (\beta'^2 - (\beta - \beta')^2)\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right)\cos\theta\, d\theta$$

which reduces to

$$F = \frac{ne^2}{D}(-\beta^2 + 2\beta\beta')$$

or

$$\frac{2I}{D}\left(\beta' - \frac{\beta}{2}\right)e.$$

Now if β is negligible compared to β', this force will be

$$\frac{2I}{D} e\,\beta'$$

which is the usual formula as obtained from the magnetic field of classic theory. For all cases of electrons in flight, the term in β is very small compared to that in β' so that it would be impossible by this means to get an experimental check on our theory.

Since any system of conductors carrying current can be treated in a manner similar to this, we may conclude that as far as this sort of force is concerned we will obtain in general the same laws for the deflection of an electron in flight in the vicinity of wires carrying current as are obtained by usual theory. This is to be expected, since our electron in flight may be considered an element of current, and we saw that our theory gave correctly the force between one complete circuit and an element of another.

We have a difference, however, when it comes to the forces acting on an electron in motion in the neighborhood of a set of static charges.

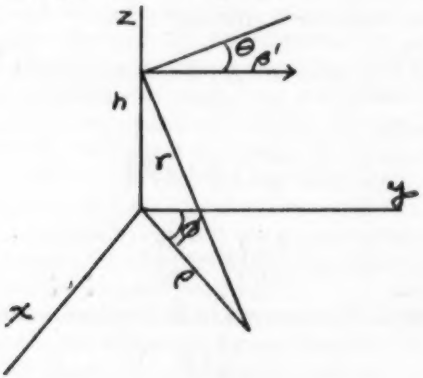
Consider two infinite parallel planes carrying charges $+q$ and $-q$ E. S. U. per unit of surface. The electron has a velocity ratio β'.

Take one plane as the xy plane, and the electron on the z axis at $z=h$, moving in the direction of y.

Take the element of surface

$$\rho d\phi d\rho$$

located by the angle ϕ and the radius ρ.



This exerts a force

$$df = \frac{qe\rho d\phi d\rho}{r^2}\left[1+\beta^2\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right)\right]$$

and the vertical component of this can be written

$$df_v = \frac{qe\rho d\phi d\rho h}{(h^2+\rho^2)\sqrt{h^2+\rho^2}}\left(1+\beta^2-\frac{3}{2}\beta^2\frac{\rho^2\cos^2\phi}{\rho^2+h^2}\right).$$

The total force then is

$$f=\int_0^\infty\int_0^{2\pi} qeh\frac{\rho d\rho}{(h^2+\rho^2)^{3/2}}\left(1+\beta^2-\frac{3}{2}\beta^2\frac{\rho^2}{\rho^2+h^2}\cos^2\phi\right)d\phi$$

$$=\int_0^\infty qeh\frac{\rho d\rho}{(h^2+\rho^2)^{3/2}}\left(2\pi(1+\beta^2)-\frac{3\pi\beta^2}{2}\frac{\rho^2}{h^2+\rho^2}\right)$$

$$=-2\pi\, qeh\,(1+\beta^2)(h^2+\rho^2)^{-1/2}$$

$$+\frac{3\pi}{2}qeh\,\beta^2\left(\rho^2+\frac{2h^2}{3}\right)(\rho^2+h^2)^{-3/2}\Big]_0^\infty$$

$$=2\pi\, qe\left(1+\frac{1}{2}\beta^2\right).$$

This is for one plane. Adding the effect of the other plane, we have

$$4\pi\, qe\left(1+\frac{1}{2}\beta^2\right).$$

An electron at rest in this field is subjected to a force

$$4\pi\, qe$$

Hence, according to our present theory, an electron in motion parallel to charged plates is acted upon by a force in the direction perpendicular to the plates and greater than that for an electron at rest by the factor

$$\left(1+\frac{1}{2}\beta^2\right).$$

We can now consider the Kaufmann-Bucherer results in detail. In this experiment was measured, using the classic notation, the radius of curvature r in an electrostatic field of intensity d,

and the radius r' in an electromagnetic field of intensity h, giving in accordance with classic theory

$$\frac{1}{r'} = \frac{e\,h}{cmv}$$

$$\frac{1}{r} = \frac{e\,a}{m\,v^2}$$

from which

$$\frac{e}{m} = \frac{c^2 d}{h^2}\frac{r}{r'^2}$$

$$v = \frac{cd}{h}\frac{r}{r'}$$

so that when r and r' are determined for the same electron we can compute v or β and the corresponding e/m. It was found, to first order terms, that e/m varied as $\left(1-\frac{1}{2}\beta^2\right)$. On the assumption of constant charge, this gives a variation of m according to $\left(1+\frac{1}{2}\beta^2\right)$ which is the relativity relation.

Let us now apply our present theory. We assume a constant mass. There is nothing to indicate for us any other assumption.

In examining expected departures from low-speed performance, let us consider the expressions for the two curvatures. We would expect, from the above, the curvature in a magnetic field to follow the low-speed law throughout. Thus, r' we would expect to follow ordinary computations from classic theory, without any correction term in β for any velocity.

On the other hand, we would expect the curvature between charged planes to be increased above that given by classic theory and constant mass by a factor $\left(1+\frac{1}{2}\beta^2\right)$.

Hence, in our expression for e/m we would expect to find r' unaltered, and r decreased by a factor $\left(1-\frac{1}{2}\beta^2\right)$. Thus we would expect to find when we measured e/m from the expression r/r'^2 an indiatcd variation according to $\left(1-\frac{1}{2}\beta^2\right)$. This is just what we do find.

It is not a simple matter of variation of charge with velocity. The reasoning can be carried through accurately only by a consideration of the electron path as dictated by the law we have used. When this is done as above, we find an agreement between our expression and the Kaufmann-Bucherer experiment.

We have considered only terms involving β^2, and not higher power terms, both in our expression for the force between moving charges and in the experimental results. Now it is probable that our force expression should contain higher power terms, and experiment is capable of measuring the β^4 term in an expression for e/m with some precision. When classic theory is used in computing the forces on fast-moving electrons it is implicitly assumed that higher power terms for force do not exist, although this does not seem a necessary assumption even there. Surely in the present formulation higher power terms are to be expected. Yet there is little basis for choice in their introduction. They may be readily chosen so as to give the β^4 term of experiments to determine e/m, but this does not completely fix the form of expression. It is to be noted in this connection that some forms of fourth power terms in the force expression will introduce a change in the second power term of the computations for electron deflections, but there are others that do not.

It is reasonable to expect the complete expression for force to be in the form of a product, one factor of which becomes zero for $\beta = 1$, such a factor for example as $(1-\beta^2 \sin^2 \theta)^{\frac{1}{2}}$, so that the force between charges becomes zero when their relative velocity is equal to the velocity of light. This is because we encounter experimentally no electron velocities above this value, and in our present formulation we are led to account for this by reason of a disappearance of the force, rather than by reason of increase of mass.

Still, with both these conditions, there is not enough data on which to choose a complete expression. We must hence leave the matter of higher power terms to be arrived at by a more complete examination of the relationship involved in the present formulation.

9. Let us now consider accelerational terms to be added to our expression for the force between moving charges, which has so far been written for uniform motion only.

The existence of induced voltages when currents are varied, in a system of circuits which are physically immobile, shows the necessity for such terms. These voltages are due to forces exerted between the individual charges present in the network, and since the induced voltages appear only when we have varying currents, that is accelerated charges, it follows that whatever our treatment may be, there must be in our expression for the force between individual charges terms involving their relative accelerations. Yet in using the classic treatment it is usually hardly realized that this is being done, for the circuit laws in terms of magnetic field appear in a manner that does not render their interpretation in terms of individual charges apparent. As a result, this aspect of classic theory has not been emphasized.

When we come to add accelerational terms to the present formulation there are several ways in which it can be done, and very little of direct experimental evidence to indicate any choice among them.

Weber[4] wrote an expression for the force between moving charges as follows:

$$f = \frac{e_1 e_2}{r^2}\left[1 + \frac{1}{c^2}\left(r\frac{\partial^2 r}{\partial t^2} - \frac{1}{2}\left(\frac{\partial r}{\partial t}\right)^2\right)\right]$$

the force to be taken as lying entirely in the line joining the charges.

This expression is not attractive as it contains a term not an inverse square term. Still, it leads correctly to all the laws of mechanical forces between complete circuits, and to induced voltages including those due to varying currents. For steady currents, if we put all accelerations equal to zero, it reduces to the expression we have been using above. There is a peculiarity here. Even with steady currents, there is a net acceleration at a bend in a circuit. These have been left out of our treatment, and in fact are neglected in the classic treatment, and yet we are in accord with experiment. Apparently the accelerations produced

[4] Elek. Maasbestimmungen, Leipzig, 1846, p. 316.

by circuit curvature have a negligible effect. The point, however, will bear further study.

Weber's expression is conservative, and that is principally why he used it. It is doubtful whether we want a conservative expression, that is an expression such that conservation of energy holds for all possible motions of charges. More attractive would be an expression such that only certain motions are conservative, the action in others involving radiation.

We may regard Weber's expression as simply one of a number of possible ones, with the advantage that all the force, including that due to acceleration, lies along the line joining the forces; and with an inverse first power term introduced in order to obtain a conservative expression.

Another way of putting in accelerational terms, which also involves an inverse first power law, is to adhere at this point to the classic treatment. The line integral expression for the mutual inductance of two circuits

$$M = \int\int \frac{ds_1 \cdot ds_2}{r}$$

suggests directly an accelerational term of the form

$$\frac{e_1 e_2}{r} \dot{\beta}$$

the force due to this term to be taken in the direction of the mutual acceleration. To this may be added any force which integrates to zero about a closed contour.

Other expressions are also possible. All of these are in accord with all the evidence we can derive from experiments on complete circuits. When it comes to the behavior of individual charges our analysis is not as yet carried to the point where we can choose between them.

In the following, therefore, we will neglect accelerational terms in the study of electron orbits. A later introduction of these will modify our present results, but apparently only slightly. On the basis of such later study it may readily become possible to discriminate between different complete formulations for the forces between moving and accelerated charges. This cannot, however, be carried through in the present paper.

10. We will now proceed to examine the behavior of a single electron about a massive nucleus, under the influence of a force given by our simple expression

$$\frac{e_1 e_2}{r^2}\left(1+\beta^2\cos^2\theta-\frac{1}{2}\beta^2\sin^2\theta\right).$$

At present we will not consider the motion of the nucleus itself, as this may readily be introduced later. Also, as noted above, we will not here consider the modifying effect of accelerational terms nor terms involving powers of β greater than the square.

We will first derive the general expression for the orbit of the electron, and compare it with Sommerfeld's results from the use of restricted relativity.

Our electron is located by polar coördinates r and ϕ. We obtain the differential equation for its motion by setting the central acceleration equal to the force given by the expression above, divided by a fixed mass m.

$$\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2=-\frac{e_1e_2}{m}\frac{1}{r^2}\left(1+\frac{r^2}{c^2}\left(\frac{d\phi}{dt}\right)^2-\frac{1}{2c^2}\left(\frac{dr}{dt}\right)^2\right).$$

Since the entire force is central, the tangential acceleration is zero.

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right)=0$$

so that we have the usual equal area law

$$r^2\frac{d\phi}{dt}=a=\text{const.}$$

Insert this in the differential equation, and use as abbreviation

$$K=\frac{e_1e_2}{m}$$

and we have

$$r^4\frac{d^2r}{dt^2}-\frac{Kr^2}{2c^2}\left(\frac{dr}{dt}\right)^2+Kr^2-a^2r+\frac{a^2K}{c^2}=0.$$

We will now alter the equation, following the usual procedure, to be in terms of u, from

$$r = \frac{1}{u}$$

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt}$$

$$\frac{d^2r}{dt^2} = -\frac{1}{u^2}\frac{d^2u}{dt^2} + \frac{2}{u^3}\left(\frac{du}{dt}\right)^2$$

obtaining

$$\frac{d^2u}{dt^2} + \left(\frac{K}{2c^2} - \frac{2}{u}\right)\left(\frac{du}{dt}\right)^2 - Ku^4 + a^2u^5 - \frac{a^2K}{c^2}u^6 = 0.$$

Also we will eliminate t by

$$\frac{du}{dt} = au^2\frac{du}{d\phi}$$

$$\frac{d^2u}{dt^2} = 2\,a^2u^3\left(\frac{du}{d\phi}\right)^2 + a^2u^4\frac{d^2u}{d\phi^2}$$

and obtain as the equation of the orbit

$$\frac{d^2u}{d\phi^2} + \frac{K}{2c^2}\left(\frac{du}{d\phi}\right)^2 = \frac{K}{c^2}u^2 - u + \frac{K}{a^2}. \tag{1}$$

In order to solve this we will remove the first derivative by using

$$q = \left(\frac{du}{d\phi}\right)^2$$

so that we obtain the linear equation

$$\frac{dq}{du} + \frac{K}{c^2}q = \frac{2K}{c^2}u^2 - 2u + \frac{2K}{a^2}$$

the solution of which is

$$q = 2u^2 - \frac{6c^2}{K}u + \frac{6c^4}{K^2} + \frac{2c^2}{a^2} + Ge^{-\frac{Ku}{c^2}}$$

where G is a constant of integration. Insert this in the above and we have

$$\frac{d^2u}{d\phi^2} = 2u - \frac{3c^2}{K} - G\frac{K}{2c^2}e^{-\frac{Ku}{c^2}}.$$

Now for hydrogen or ionized helium, at least, $\frac{Ku}{c^2}$ is so small, about 10^{-4}, that we may write this expression very approximately

$$\frac{d^2u}{d\phi^2} = 2u - \frac{3c^2}{K} - G\frac{K}{2c^2}\left(1 - \frac{Ku}{c^2}\right). \tag{2}$$

To determine G we will set $\frac{du}{d\phi}$ equal to zero and solve (1) and (2) simultaneously giving

$$G = -\frac{4c^4}{K^2}\frac{u_0 - \frac{3c^2}{2K}}{u_0 - \frac{c^2}{K}}$$

where

$$u_0 = \frac{c^2}{2K} - \frac{1}{2}\sqrt{\frac{c^4}{K^2} - \frac{4c^2}{a^2}}.$$

To the same approximation as before this may be written

$$G = -\frac{2c^4}{K^2}\left(3 + \frac{Ku_0}{c^2}\right)$$

and since $\frac{K^2}{c^2a^2}$ is also small

$$u_0 = \frac{K}{a^2} - \frac{K^3}{c^2a^4}$$

so that the equation may be written

$$\frac{d^2u}{d\phi^2} = 2u - \frac{3c^2}{K} + \frac{c^2}{K}\left[3 + \frac{K}{c^2}\left(\frac{K}{a^2} - \frac{K^3}{c^2a^4}\right)\right]\left[1 - \frac{Ku}{c^2}\right]$$

or

$$\frac{d^2u}{d\phi^2} = -u\left(1 - \frac{K^2}{c^2a^2}\right) + \frac{K}{a^2} - \frac{K^3}{a^4c^2}.$$

The solution of this equation is

$$u = A \sin \alpha\phi + B \cos \alpha\phi + D$$

where α is given by

$$\alpha^2 = 1 - \frac{K^2}{c^2 a^2} = 1 - \frac{e_1^2 e_2^2}{c^2 a^2 m^2}.$$

The orbit is hence a rosette or precessing ellipse, the rate of precession being given by α. This is just the form of orbit obtained by Sommerfeld[5] on the basis of restricted relativity and a variant mass, and from which followed his theory of fine structure. We obtain this same result from our law of force between moving charges, and an invariant mass. The constant α is exactly the same as Sommerfeld's, and to the same extent in accord with experiment.[6]

The result still needs further examination in view of terms of higher powers of β and accelerational terms. It is quite possible that these will yield closer accord with experiment even than we now have. At any rate the present theory appears to yield in its simple form much the same results as are obtained by restricted relativity, while being free from the difficulties in which the latter is at present involved.

11. So far we have been concerned only with the parallelism of classic theory and the present formulation. There is one important matter, however, on which we can immediately indicate the possibility of progress beyond that given by the classic treatment. This concerns the nature of the nucleus of an atom.

From Equation (1) above it is readily apparent that circular orbits are possible as a particular solution of the equation. Let us examine a condition for their existence. Consider an electron at P moving with vertical velocity ratio β.

Force is:

$$\frac{e_1 e_2}{r^2}(1+\beta^2)$$

and acceleration, directed toward O, is:

$$a = \frac{e_1 e_2}{m r^2}(1+\beta^2).$$

[5] A. Sommerfeld, Annalen d. Phys. 1916, p. 53 or Atombau v. Spektrallinien, 1922 Braunschweig.

[6] Vallarta. Sommerfeld's Theory of Fine Structure from the Standpoint of General Relativity, Jour. of Math. & Phys., Apr. 1925.

Now suppose we choose β in such a manner that this acceleration is equal to

$$a = \frac{v^2}{r} = \frac{\beta^2 c^2}{r}.$$

That is

$$\frac{v^2}{r} = \frac{e_1 e_2}{m r^2}\left(1 + \frac{v^2}{c^2}\right)$$

or

$$v^2\left(\frac{1}{r} - \frac{e_1 e_2}{m r^2 c^2}\right) = \frac{e_1 e_2}{m r^2}$$

or

$$v = \sqrt{\frac{e_1 e_2 c^2}{mrc^2 - e_1 e_2}}.$$

Then the acceleration will be v^2/r directed toward O. But a particle with this central acceleration will move in a circle with center O and radius r, and since the above conditions are satisfied at each point of its path, it will continue to so move.

There will be a real value of v if

$$r > \frac{e_1 e_2}{m c^2}$$

but not if r is less than this value.

There is therefore a critical radius only outside of which are circular orbits possible, and approximately

$$r = \frac{1}{(2.1\times 10^9)^2 (3\times 10^{10})^2 \times 9\times 10^{-28}} \text{ cm.}$$

$$r = 2.8\times 10^{-13} \text{ cm.}$$

or the critical diameter will be

$$5\times 10^{-13} \text{ cm.}$$

Now, the experimental determinations of the diameter of the nucleus of an atom give values between 10^{-12} and 10^{-13} cm.

Outside this critical radius circular orbits, and the general precessing elliptical orbit already examined, are possible.

Inside this radius, we have a peculiar affair. Whenever in the motion of an electron its path becomes tangent to a circle with the positive charge as center, then, no matter what value of velocity it may have, the curvature of its path is greater than that of the central circle through the same point, so that it will pass inside this circle. Hence, inside this critical radius the electron apparently spirals in toward the positive charge continuously. But its speed also increases to high values. Hence, before we can consider this limit of the motion we need the rest of the series for our expression for force, not merely the terms in β^2.

These extra terms, however, will not affect the existence of a critical diameter, of about the value of the observed nuclear diameter, within which an electron will be imprisoned under a motion far different from that which applies outside the critical diameter. Just what happens to an electron which drops within this critical distance of a positive charge, and which cannot thereafter escape, at least if alone, is another matter.

We may speculate, however, that if, upon adding the complete series of terms in β, the force between charges disappears for $\beta=1$, then the velocity of light will never be exceeded by the electron in its flight, the electrons in the nucleus will settle to orbits with velocities less than c, which is in accord with our observations that electrons leaving the nucleus do so with velocities approaching but not exceeding the velocity of light.

12. It thus appears that variant charge theory offers a parallel line of attack upon atomistics which should be considered as carefully as classic theory and variant mass have been treated. The immediate objections to this parallel attack do not appear formidable. It will give at least some of the results that have been regarded as successes for restricted relativity, and it holds the promise of being capable of even more

THE HARMONIC ANALYSIS OF IRREGULAR MOTION

Second Paper

By NORBERT WIENER

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Introduction

In a previous paper by the same name, also published in this Journal, the author gave a method of analyzing into its harmonic components a function neither periodic, as are the functions representable by Fourier series, nor asymptotically zero, as are the functions for which the Fourier integral is the appropriate development. The conditions to be fulfilled by a function $f(t)$ possessing such an analysis are:

(*a*) that its mean square value shall exist;

(*b*) that the mean square values of a certain denumerable set of its transforms

$$\int_{-\infty}^{\infty} \Phi(t-\tau) f(\tau) d\tau \qquad [\Phi \text{ even}]$$

shall exist;

(*c*) that the mean square value of the transform shall approach the mean square value of $f(t)$ as

$$\frac{2}{\pi}\int_0^\infty \Phi(t)\cos t\,u\,dt$$

approaches 1.

It is the purpose of the present paper to exhibit a function $f(x)$ of this sort, which in addition has in the language of the previous paper a continuous spectrum. In this way, we shall not only supplement the results already obtained by an existence theory, but shall also contribute to the rigorous mathematical theory of the disturbance of a linear oscillating system by haphazard disturbances, as exemplified by the theory of tube noises in radiotelephony. Let it be noted that our existence theory is dependent throughout on the multiplicative axiom of Zermelo. We shall assume this axiom without explicit reference in all that follows.

§1. The distribution of functions over an infinite range.

In an earlier paper,[1] the author has discussed the distribution of a class of functions defined for arguments between $t=0$ and $t=1$, which have as a class the following properties:

(*a*) The change in the value of a function between $t=t_1$ and $t=t_2$ is independent of the value of the function at and before $t=t_1$;

(*b*) The change in question has a Gaussian distribution;

(*c*) The parameter of this distribution depends only on t_2-t_1.

We shall call a class of functions with the probability distribution indicated in that paper a set *normally distributed* over the time-interval (0, 1). We wish here to generalize the notion of a normal distribution to functions defined over the time-interval $(-\infty, \infty)$.

Let $f(t)$ be a function of a normally distributed class over the interval (0, 1). Let us form the function

$$f\left(\frac{1}{\pi}\tan^{-1}t+\frac{1}{2}\right)$$

[1] Differential-Space, this Journal, Vol. 2 (1923), pp. 131–174.

This will be defined over the whole infinite set of values of t, and will furthermore satisfy conditions (a) and (b). It will not, however, satisfy condition (c), since the interval (t_1, t_2) will be drawn out into the interval $(-\cot \pi t_1, -\cot \pi t_2)$.

We can, however, compensate for this change in the range of arguments of the function by a corresponding change in the values of the function. We should perhaps expect that $g(t)$ would be uniformly distributed if

$$g(t) = \int_0^t \frac{1}{\frac{d}{dt}\left(\frac{1}{\pi}\tan^{-1}t + \frac{1}{2}\right)}\, df\left(\frac{1}{\pi}\tan^{-1}t + \frac{1}{2}\right)$$

$$= \pi \int^t (1+t^2)\, df\left(\frac{1}{\pi}\tan^{-1}t + \frac{1}{2}\right).$$

It must be remembered, however, that in such a haphazard motion as we are discussing, the change of g is proportional for a small interval of time, not to the change in t, but roughly to the change in the square root of t. This would give us formally

$$g(t) = \sqrt{\pi}\int_0^t \sqrt{1+t^2}\, df\left(\left(\frac{1}{\pi}\right)\tan^{-1}t + \frac{1}{2}\right).$$

Since, however, f is practically never of limited total variation, we must change this definition into a form where it makes sense. To do this, we integrate formally by parts, obtaining.

$$g(t) = \sqrt{\pi}\int_{-\infty}^{\infty} f\left(\left(\frac{1}{\pi}\right)\tan^{-1}\xi + \frac{1}{2}\right) d\phi_t(\xi)$$

$$= \sqrt{\pi}\int_0^1 f(u)\, d\psi_t(u), \tag{1}$$

where

$$\left.\begin{aligned} &\phi_t(\xi) = \sqrt{1+\xi^2} && [0 \le \xi \le t]; \\ &\phi_t(\xi) = 0 && [0 > \xi \text{ or } t < \xi]; \end{aligned}\right\} [t > 0]$$

$$\left.\begin{aligned} &\psi_t(u) = \csc \pi u && \left[\frac{1}{2} \le u \le \frac{\tan^{-1}t}{\pi} + \frac{1}{2}\right]; \\ &\psi_t(u) = 0 && \left[\frac{1}{2} \ge u \text{ or } u \ge \frac{\tan^{-1}t}{\pi} + \frac{1}{2}\right]; \end{aligned}\right\} [t > 0]$$

$$\left.\begin{aligned}&\phi_t(\xi)=\sqrt{1+\xi^2} && [t\leq\xi\leq 0];\\ &\phi_t(\xi)=0 && [\xi>0 \text{ or } \xi<t];\end{aligned}\right\}\ [t<0]$$

$$\left.\begin{aligned}&\psi_t(u)=\csc \pi u && \left[\frac{1}{2}\geq u\geq\frac{\tan^{-1}t}{\pi}+\frac{1}{2}\right];\\ &\psi_t(u)=0 && \left[\frac{1}{2}\leq u \text{ or } u\leq\frac{\tan^{-1}t}{\pi}+\frac{1}{2}\right];\end{aligned}\right\}\ [t<0]$$

$$g(0)=0. \tag{2}$$

Using the results of an earlier paper,[3] we obtain for the average value of $g(t_1)\ g(t_2)$,

$$\begin{aligned}&\frac{\pi\gamma}{2}\int_0^1 d\psi_{t_1}(u)\int_0^u v\,d\psi_{t_2}(v)+\frac{\pi^2\gamma}{2}\int_0^1 d\psi_{t_2}(u)\int_0^u v\,d\psi_{t_1}(v)\\ &\quad=-\frac{\pi\gamma}{2}\int_0^1 u[\psi_{t_1}(u)d\psi_{t_2}(u)+\psi_{t_2}(u)d\psi_{t_1}(u)]\\ &\quad=-\frac{\pi\gamma}{2}\int_0^1 u\,d[\psi_{t_1}(u)\psi_{t_2}(u)]\\ &\quad=\frac{\pi\gamma}{2}\int_0^1 \psi_{t_1}(u)\psi_{t_2}(u)\,du\end{aligned}$$

$$\left\{\begin{aligned}&=\frac{\pi\gamma}{2}\int_{1/2}^{\frac{1}{\pi}\tan^{-1}t_1+1/2}\csc^2\pi u\,du; && \left[\begin{matrix}0<t_1\leq t_2\\ \text{or } 0>t_1\geq t_2\end{matrix}\right]\\ &=\frac{\pi\gamma}{2}\int_{1/2}^{\frac{1}{\pi}\tan^{-1}t_2+1/2}\csc^2\pi u\,du; && \left[\begin{matrix}0<t_2\leq t_1\\ \text{or } 0>t_2\geq t_1\end{matrix}\right]\\ &=0 \quad ; && [t_1t_2\leq 0]\end{aligned}\right.$$

$$\left\{\begin{aligned}&=\gamma t_1/2 \quad ; && \left[\begin{matrix}0<t_1\leq t_2\\ \text{or } 0>t_1\geq t_2\end{matrix}\right]\\ &=\gamma t_2/2 \quad ; && \left[\begin{matrix}0<t_2\leq t_1\\ \text{or } 0>t_2\geq t_1\end{matrix}\right]\\ &=0 \quad . && [t_1\,t_2\leq 0]\end{aligned}\right. \tag{3}$$

[3] Cf. *The Average of an Analytic Function*, Proc. Nat. Acad. Sci. 1921, p. 256.

Here r is a parameter determining the mean square variation of $f(t)$ over a given time.

We thus obtain the same formal expression for the mean of $g(t_1)g(t_2)$ as for the mean of $f(t_1)f(t_2)$, which is as it should be. We furthermore see that the mean of $g(t_1)(g(t_2)-g(t_1))$ is 0, thus showing that there is at least a linear independence between $g(t_1)$ and $g(t_2)-g(t_1)$. The mean square value of $g(t_2)-g(t_1)$ is always $r|t_2-t_1|/2$, which is in harmony with condition (c).

§2. Independent linear functionals.

In the author's earlier paper, the notion of independent linear functionals was introduced. It was there shown that if

$$\int_x^1 K(x)dx \qquad \text{and} \qquad \int_x^1 Q(x)dx$$

are normal and orthogonal, and $\phi(u)$ is bounded and uniformly continuous, the mean of

$$\phi\left(\int_0^1 K(x)f(x)dx, \quad \int_0^1 Q(x)f(x)dx\right)$$

with respect to f is the same as the mean of

$$\phi\left(\int_0^1 K(x)f_1(x)dx \quad \int_0^1 Q(x)f_2(x)dx\right)$$

with respect to f_1 and f_2. This notion may be extended to a set containing any finite number of independent functionals. It may also be extended to linear functionals of the form

$$\int_0^1 f(x)d\,\alpha(x)$$

Without any fundamental change, the argument of the author's earlier paper will show that if

$$\alpha(1)-\alpha(x) \qquad \text{and} \qquad \beta(1)-\beta(x)$$

are normal and orthogonal, and ϕ is bounded and uniformly continuous, the mean of

$$\phi\left(\int_0^1 f(x)d\,\alpha(x), \quad \int_0^1 f(x)d\,\beta(x)\right)$$

with respect to f is the same as the mean of

$$\phi\left(\int_0^1 f_1(x)d\,\alpha(x),\ \int_0^1 f_2(x)d\,\beta(x)\right)$$

with respect to f_1 and f_2. The extension to any finite set of independent linear functionals also holds. It follows at once that the mean of

$$\phi(g(t_2)-g(t_1),\quad g(t_1)-g(t_0)) \qquad [t_0<t_1<t_2]$$

is the same as the mean of

$$\phi(g_1(t_2)-g_1(t_1),\quad g_2(t_1)-g_2(t_0))$$

with respect to the two normally distributed functions $f_1(t)$ and $f_2(t)$ on which g_1 and g_2 respectively depend. This is in accordance with (*a*).

Let us now consider two expressions of the form

$$\int_{-\infty}^{\infty} K(x)g(x)dx, \qquad \int_{-\infty}^{\infty} Q(x)g(x)dx,$$

where $K(x)\sqrt{1+x^2}$ and $Q(x)\sqrt{1+x^2}$ are absolutely integrable over the whole infinite line. We have (except in the case where $g(x)\neq O(x)$ at infinity, which is of 0 probability)

$$\begin{aligned}\int_{-\infty}^{\infty} K(x)g(x)dx &= \sqrt{\pi}\int_{\infty}^{\infty} K(x)dx\int_0^1 f(u)d\psi_x(u)\\ &= \sqrt{\pi}\int_0^1 f(u)d_u\int_{-\infty}^{\infty} K(x)\psi_x(u)dx\\ &= \sqrt{\pi}\int_{\frac{1}{2}}^1 f(u)d_u\int_{-\cot \pi u}^{\infty} K(x)\csc\,\pi u\,dx\\ &\quad -\sqrt{\pi}\int_0^{\frac{1}{2}} f(u)d_u\int_{-\infty}^{-\cot \pi u} K(x)\csc\,\pi u\,dx.\end{aligned}$$

Now, we have

$$\pi\int_0^1 du\int_{-\infty}^{\infty}K(x)[\psi_x(1)-\psi_x(u)]dx\int_{-\infty}^{\infty}Q(x)[\psi_x(1)-\psi_x(u)]dx$$

$$=\pi\int_{1/2}^{1}du\int_{-\cot \pi u}^{\infty}K(x)\csc \pi u\,dx\int_{-\cot \pi u}^{\infty}Q(y)\csc \pi u\,dy$$

$$+\pi\int_0^{1/2}du\int_{-\infty}^{-\cot \pi u}K(x)\csc \pi u\,dx\int_{-\infty}^{-\cot \pi u}Q(y)\csc \pi u\,dy$$

$$=\int_{-\infty}^{0}dw\int_{w}^{\infty}K(x)dx\int_{w}^{\infty}Q(y)dy$$

$$+\int_0^{\infty}dw\int_{-\infty}^{w}K(x)dx\int_{-\infty}^{w}Q(y)dy. \tag{4}$$

That is, we have the following theorem: *let* $K(x)\sqrt{1+x^2}$ and $Q(x)\sqrt{1+x^2}$ *be absolutely integrable over the whole infinite line, and let*

$$\int_{-\infty}^{\infty}K(x)dx=\int_{-\infty}^{\infty}Q(x)dx=0.$$

Let

$$\int_{-\infty}^{\infty}dw\int_{-\infty}^{w}K(x)dx\int_{-\infty}^{w}Q(y)dy=0;$$

$$\int_{-\infty}^{\infty}dw\int_{-\infty}^{w}K(x)dx\int_{-\infty}^{w}K(y)dy=\int_{-\infty}^{\infty}dw\int_{-\infty}^{w}Q(x)dx\int_{-\infty}^{w}Q(y)dy=1.$$

Then

$$\int_{-\infty}^{\infty}K(x)g(x)dx \quad \textit{and} \quad \int_{-\infty}^{\infty}Q(x)g(x)dx$$

will be independent functionals, with mean square values r^2. A similar result will hold with regard to any finite set of such functionals.

§3. The existence of the mean square of the transform of a function.

Now let us consider the expression

$$\frac{1}{2A}\int_{-A}^{A}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2dt \tag{5}$$

where

$$\Phi(\tau)=O(\tau^{-3}) \text{ at } \pm\infty\,; \quad \int_{-\infty}^{\infty}\Phi(\tau)d\tau=0. \tag{6}$$

We have

$$\frac{1}{2A}\int_{-A}^{A}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2dt$$

$$=\int_{-\infty}^{\infty}g(\sigma)d\sigma\int_{-\infty}^{\infty}g(\tau)d\tau\left[\frac{1}{2A}\int_{-A}^{A}\Phi(\sigma-t)\Phi(\tau-t)dt\right]$$

$$=\int_{-\infty}^{\infty}g(\sigma)d\sigma\int_{-\infty}^{\infty}g(\tau)F_A(\sigma,\tau)d\tau. \tag{7}$$

Let us put

$$G_A(\sigma,\tau)=\int_{-\infty}^{\sigma}d\alpha\int_{-\infty}^{\tau}F_A(\alpha,\beta)d\beta. \tag{8}$$

Let us now consider the Fredholm equation

$$u(\sigma)=\lambda\int_{-\infty}^{\infty}G_A(\sigma,\tau)u(\tau)d\tau \tag{9}$$

This will be satisfied for a denumerable set of values λ_n of λ, and for a denumerable set of corresponding functions $u_n(\tau)$. These can be chosen in such a way that

$$\int_{-\infty}^{\infty}u_m(\tau)u_n(\tau)d\tau=\delta_{mn}. \tag{10}$$

We shall have at least formally

$$G_A(\sigma,\tau)=\sum_1^{\infty}\frac{u_n(\sigma)u_n(\tau)}{\lambda_n} \tag{11}$$

and hence again formally

$$F_A(\sigma,\tau)=\sum_1^{\infty}\frac{u_n'(\sigma)u_n'(\tau)}{\lambda_n}. \tag{12}$$

If this latter formula holds, it follows that

$$\int_{-\infty}^{\infty}g(\sigma)d\sigma\int_{-\infty}^{\infty}g(\tau)F_A(\sigma,\tau)d\tau$$

$$=\sum_1^{\infty}\frac{1}{\lambda_n}\left[\int_{-\infty}^{\infty}g(\sigma)u_n'(\sigma)d\sigma\right]^2. \tag{13}$$

Then

$$\frac{1}{2A}\int_{-A}^{A}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)d\tau\int_{-\infty}^{\tau}u_m(\alpha)d\alpha\right]^2dt$$

$$=\sum_1^{\infty}\frac{1}{\lambda_n}\left[\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\sigma}u_m(\alpha)d\alpha\right)u_n'(\sigma)d\sigma\right]^2$$

$$=\sum_1^{\infty}\frac{1}{\lambda_n}\left[u_n(\infty)\int_{-\infty}^{\infty}u_m(\alpha)d\alpha-\int_{-\infty}^{\infty}u_m(\sigma)u_n(\sigma)d\sigma\right]^2=\frac{1}{\lambda_m}. \tag{14}$$

That is, every λ_m is positive. In this proof we have used the obvious consequences of our hypotheses, that $G_A(\sigma, \tau)$ is $O\left(\frac{1}{\sigma^2+\tau^2}\right)$ at $\sigma, \tau=\pm\infty$, and that $u_n(\infty)=0$. From the latter fact, assuming that our bilinear formula holds, it results that we have reduced

$$\int_{-\infty}^{\infty}g(\sigma)d\sigma\int_{-\infty}^{\infty}g(\tau)F_A(\sigma,\tau)d\tau$$

to a sum of squares of independent linear functionals.

We have

$$\int_{-\infty}^{\infty}[u_n(\sigma)]^2d\lambda=\lambda_n\int_{-\infty}^{\infty}u_n(\sigma)d\sigma\int_{-\infty}^{\infty}G_A(\sigma,\tau)u(\tau)d\tau$$

$$=\frac{\lambda_n}{2A}\int_{-\infty}^{\infty}u_n(\sigma)d\sigma\int_{-\infty}^{\infty}u_n(\tau)d\tau\int_{-A}^{A}dt\int_{-\infty}^{\sigma}\Phi(\alpha-t)d\alpha\int_{-\infty}^{\tau}\Phi(\beta-t)d\beta$$

$$=\frac{\lambda_n}{2A}\int_{-\infty}^{\infty}d\mu\int_{-\infty}^{\infty}u_n(\sigma)u_n(\sigma+\mu)d\sigma\int_{-A}^{A}dt\int_{-\infty}^{\sigma}\Phi(\alpha-t)d\alpha$$
$$\int_{-\infty}^{\sigma+\mu}\Phi(\beta-t)d\beta$$

$$\leq\frac{\lambda_n}{2A}\int_{-\infty}^{\infty}[u_n(\sigma)]^2d\sigma\int_{-\infty}^{\infty}d\mu\left|\int_{-A+\sigma}^{A+\sigma}dt\int_{-\infty}^{0}\Phi(\alpha-t)d\alpha\int_{-\infty}^{\mu}\Phi(\beta-t)d\beta\right|$$

$$\leq\frac{\lambda_n}{2A}\int_{-\infty}^{\infty}[u_n(\sigma)]^2d\sigma\int_{-\infty}^{\infty}d\mu\int_{-\infty}^{\infty}dt\,\frac{C}{1+t^2}\cdot\frac{C}{1+(\mu-t)^2} \tag{15}$$

where C is some constant. Inasmuch as the latter integral converges, we have a quantity R such that $\lambda_n \geq RA$.

We have now the task of establishing (13) rigorously. Owing to the continuity of the functionals involved, it is enough to establish (13) for differentiable functions $g(\alpha)$ which are $O(\sqrt{\alpha})$ at infinity. Under these conditions, we see on integrating by parts that (13) is equivalent to

$$\int_{-\infty}^{\infty} g'(\sigma)d\sigma \int_{-\infty}^{\infty} g'(\tau)G_A(\sigma, \tau)d\tau = \sum_1^{\infty} \frac{1}{\lambda_n}\left[\int_{-\infty}^{\infty} g'(\sigma)u_n(\sigma)d\sigma\right]^2. \tag{16}$$

This is, however, an immediate consequence of (11), provided the latter formula holds uniformly, or indeed, if this latter formula holds in the sense of convergence in the mean. On the hypothesis that all the λ_n's are positive, the analogue of (11) has been proved by Mercer[3] to hold uniformly and absolutely in the case where the interval of integration is finite. If the interval is infinite, our hypotheses (6) allow us to transform it into a finite interval in such a manner that Mercer's theorem holds everywhere, except possibly in the neighborhood of the end points. If we replace uniform convergence by convergence in the mean, this latter difficulty is obviated.

The expression

$$\sum_1^N \frac{1}{\lambda_n}\left[\int_{-\infty}^{\infty} g(\sigma)u_n'(\sigma)d\sigma\right]^2$$

will have a mean value. It is true that the square of a number is not a uniformly continuous function of that number over an infinite range, but it is the upper bound of an increasing sequence of such functions, and hence[4] will have a mean value. The computation of this value is easy: it will be

$$r^2 \sum_1^N \frac{1}{\lambda_n},$$

[3] T. Mercer. Functions of Positive and Negative Type and their Connection with the Theory of Integral Equations. Phil. Trans. Royal. Soc. (A) Vol. 209, pp. 415–446. 1909.

[4] Cf. Differential-Space, §§9 and 10; The Average of an Analytic Functional.

in accordance with our fundamental theorem concerning independent functionals. If we use again our fundamental theorem concerning increasing sequences of functionals, to the effect that if we have an increasing sequence of functionals whose means approach a limit, this will be the mean of the limit functional of the sequence, we see that the mean of

$$\frac{1}{2A}\int_{-A}^{A}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2 dt$$

will be

$$\begin{aligned}
r^2\sum_1^\infty\frac{1}{\lambda_n} &= r^2\int_{-\infty}^{\infty}G_A(\sigma,\sigma)d\sigma \\
&= \frac{r^2}{2A}\int_{-\infty}^{\infty}d\sigma\int_{-\infty}^{\sigma}d\alpha\int_{-\infty}^{\sigma}d\beta\int_{-A}^{A}\Phi(\alpha-t)\Phi(\beta-t)dt \\
&= \frac{r^2}{2A}\int_{-A}^{A}dt\int_{-\infty}^{\infty}d\sigma\left[\int_{-\infty}^{\sigma}\Phi(\alpha-t)d\alpha\right]^2 \\
&= r^2\int_{-\infty}^{\infty}d\sigma\left[\int_{-\infty}^{\sigma}\Phi(\alpha)d\alpha\right]^2. \qquad (17)
\end{aligned}$$

It will be noted that this is independent of A.

An argument of the same type will show the existence of the mean of

$$\left\{\frac{1}{2A}\int_{-A}^{A}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2 dt\right\}^2.$$

This may be written

$$\left\{\sum_1^\infty\frac{1}{\lambda_n}\left[\int_{-\infty}^{\infty}g(\sigma)u_n'(\sigma)d\sigma\right]^2\right\}^2$$

$$= \lim_{N\to\infty}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{1}{\lambda_j\lambda_k}\left[\int_{-\infty}^{\infty}g(\sigma)u_j'(\sigma)d\sigma\right]^2\left[\int_{-\infty}^{\infty}g(\sigma)u_k'(\sigma)d\sigma\right]^2. \qquad (18)$$

The means of the several terms in this sum have already been evaluated by the author,[8] yielding us the result for the mean of this expression

$$\left[r^2\sum_1^\infty \frac{1}{\lambda_n}\right]^2+2r^4\sum_1^\infty\frac{1}{\lambda_n^2}. \tag{19}$$

Using the distributive property of the mean, we see that

$$\begin{aligned}&\left\{\frac{1}{2A}\int_{-A}^{A}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2dt-r^2\sum_1^\infty\frac{1}{\lambda_n}\right\}^2\\ &=\left\{\frac{1}{2A}\int_{-A}^{A}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2dt\right\}^2\\ &-2r^2\sum_1^\infty\frac{1}{\lambda_n}\cdot\frac{1}{2A}\int_{-A}^{A}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2dt\\ &+r^4\left[\sum_1^\infty\frac{1}{\lambda_n}\right]^2\end{aligned} \tag{20}$$

has the mean

$$2r^4\sum_1^\infty\frac{1}{\lambda_n^2}\leq\frac{2r^4}{RA}\int_{-\infty}^{\infty}d\sigma\left[\int_{-\infty}^{\sigma}\Phi(\alpha)d\alpha\right]^2=B/A \tag{21}$$

where B is some constant. In a similar manner, we may obtain for the mean of

$$\left\{\frac{1}{2A}\int_{-A}^{A}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2dt-r^2\sum_1^\infty\frac{1}{\lambda_n}\right\}^4$$

the expression

$$48r^8\sum_1^\infty\frac{1}{\lambda_n^4}+24r^8\left[\sum_1^\infty\frac{1}{\lambda_n^2}\right]^2 \tag{22}$$

which is less than some expression C/A^2. The argument that

$$\begin{aligned}&\text{mean}\left\{\frac{1}{2A}\int_{-A}^{A}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2dt-r^2\sum_1^\infty 1/\lambda_n\right\}^4\\ &=\lim_{N\to\infty}\text{mean}\left\{\sum_1^N\frac{1}{\lambda_n}\left[\int_{-\infty}^{\infty}g(\sigma)u_n'(\sigma)d\sigma\right]^2-r^2\sum^N 1/\lambda_n\right\}^4\end{aligned}$$

[8] The Average of an Analytic Functional. Also Differential-Space, §9.

follows exactly the same lines as in the second degree case. Let us use Σ' to denote a sum for several *distinct* subscripts. We have

$$\text{mean}\left\{\sum_1^N \frac{1}{\lambda_n}\left[\left(\int_{-\infty}^{\infty} g(\sigma)u_n'(\sigma)d\sigma\right)^2 - r^2\right]\right\}^4$$

$$= \text{mean}\sum_1^N \frac{1}{\lambda_n^4}\left[\left(\int_{-\infty}^{\infty} g(\sigma)u_n'(\sigma)d\sigma\right)^2 - r^2\right]^4$$

$$+4\ \text{mean}\sum_{m,n=1}^{N}{}' \frac{1}{\lambda_m^3\lambda_n}\left[\left(\int_{-\infty}^{\infty} g(\sigma)u_m'(\sigma)d\sigma\right)^2 - r^2\right]^3 \left[\left(\int_{-\infty}^{\infty} g(\sigma)u_n'(\sigma)d\sigma\right)^2 - r^2\right]$$

$$+\ 6\ \text{mean}\sum_{m,n=1}^{N}{}' \frac{1}{\lambda_m^2\lambda_n^2}\left[\left(\int_{-\infty}^{\infty} g(\sigma)u_m'(\sigma)d\sigma\right)^2 - r^2\right]^2 \left[\left(\int_{-\infty}^{\infty} g(\sigma)u_n'(\sigma)d\sigma\right)^2 - r^2\right]^2$$

$$+12\ \text{mean}\sum_{m,n,p=1}^{N}{}' \frac{1}{\lambda_m^2\lambda_n\lambda_p}\left[\left(\int_{-\infty}^{\infty} g(\sigma)u_m'(\sigma)d\sigma\right)^2 - r^2\right]^2 \left[\left(\int_{-\infty}^{\infty} g(\sigma)u_n'(\sigma)d\sigma\right)^2 - r^2\right] \cdot \left[\left(\int_{-\infty}^{\infty} g(\sigma)u_p'(\sigma)d\sigma\right)^2 - r^2\right]$$

$$+24\ \text{mean}\sum_{m,n,p,q=1}^{N}{}' \frac{1}{\lambda_m\lambda_n\lambda_p\lambda_q}\left[\left(\int_{-\infty}^{\infty} g(\sigma)u_m'(\sigma)d\sigma\right)^2 - r^2\right] \left[\left(\int_{-\infty}^{\infty} g(\sigma)u_n'(\sigma)d\sigma\right)^2 - r^2\right] \cdot \left[\left(\int_{-\infty}^{\infty} g(\sigma)u_p'(\sigma)d\sigma\right)^2 - r^2\right] \left[\left(\int_{-\infty}^{\infty} g(\sigma)u_q'(\sigma)d\sigma\right)^2 - r^2\right]$$

$$= \sum_1^N \frac{1}{\lambda_n^4} \left\{ \text{mean}\left(\int_{-\infty}^{\infty} g(\sigma) u_n'(\sigma) d\sigma \right)^4 \right.$$
$$-4\, r^2\, \text{mean}\left(\int_{-\infty}^{\infty} g(\sigma) u_n'(\sigma) d\sigma \right)^6$$
$$+6\, r^4\, \text{mean}\left(\int_{-\infty}^{\infty} g(\sigma) u_n'(\sigma) d\sigma \right)^4$$
$$\left. -4\, r^6\, \text{mean}\left(\int_{-\infty}^{\infty} g(\sigma) u_n'(\sigma) d\sigma \right)^2 + r^8 \right\}$$

$$+6 \sum_{m,n=1}^{N}{}' \frac{1}{\lambda_m^2 \lambda_n^2} \left\{ \text{mean}\left(\int_{-\infty}^{\infty} g(\sigma) u_m'(\sigma) d\sigma \right)^4 \right.$$
$$\left. -2\, r^2\, \text{mean}\left(\int_{-\infty}^{\infty} g(\sigma) u_m'(\sigma) d\sigma \right)^2 + r^4 \right\}$$
$$\left\{ \text{mean}\left(\int_{-\infty}^{\infty} g(\sigma) u_n'(\sigma) d\sigma \right)^4 \right.$$
$$\left. -2\, r^2\, \text{mean}\left(\int_{-\infty}^{\infty} g(\sigma) u_n'(\sigma) d\sigma \right)^2 + r^4 \right\}$$

$$= \sum_1^N \frac{r^8}{\lambda_n^4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\xi^8 - 4\xi^6 + 6\xi^4 - 4\xi^2 + 1) e^{-\frac{\xi^2}{2}} d\xi$$

$$+6 \sum_{m,n=1}^{N}{}' \frac{r^8}{\lambda_m^2 \lambda_n^2} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\xi^4 - 2\xi^2 + 1) e^{-\frac{\xi^2}{2}} d\xi \right]^2$$

$$= 60\, r^8 \sum_1^N \frac{1}{\lambda_n^4} + 24\, r^8 \sum_{m,n=1}^{N}{}' \frac{1}{\lambda_m^2 \lambda_n^2}$$

$$= 48\, r^8 \sum_1^N \frac{1}{\lambda_n^4} + 24\, r^8 \left[\sum_1^N \frac{1}{\lambda_n^2} \right]^2.$$

Let us now consider a functional of f which is P^4 whenever

$$\left\{ \frac{1}{2A} \int_{-A}^{A} \left[\int_{-\infty}^{\infty} \Phi(\tau - t) g(\tau) d\tau \right]^2 dt - r^2 \sum_1^{\infty} \frac{1}{\lambda_n} \right\}^4 \geq P^4 \qquad (23)$$

and which is zero otherwise. It is clear that this functional never exceeds

$$\left\{\frac{1}{2A}\int_{-A}^{A}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2dt-r^2\sum_1^\infty\frac{1}{\lambda_n}\right\}^4 \tag{24}$$

Hence the lower bound of the mean of larger functionals, or, as we call it, the upper mean of this functional, cannot exceed C/A^2. We term the upper mean of the functional which is 1 on a given class of functions and 0 elsewhere, the outer probability that a function belongs to this class. It follows that the outer probability that a function $g(\tau)$ has the property (23) is not greater than C/A^2P^4. There is, moreover, a theorem due to the fact that our mean is a Daniell integral,[6] to the effect that the outer probability of the disjunction of two events cannot exceed the sum of their outer probabilities.

It follows that the outer probability that some expression of the form

$$\left|\frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2dt-r^2\sum_1^\infty\frac{1}{\lambda_n}\right| \qquad [A\leq\alpha\leq A+1]$$

should exceed $3P$ cannot exceed the combined outer probabilities that

$$\left|\frac{1}{2(A+1)}\int_{-A-1}^{A+1}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2dt-r^2\sum_1^\infty\frac{1}{\lambda_n}\right|\geq P;$$

$$\int_{A}^{A+1}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2dt\geq 2P(A+1);$$

$$\int_{-A-1}^{A}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2dt\geq 2P(A+1); \tag{25}$$

and hence does not exceed some expression S/A^2. To prove this, it is only necessary to carry through an argument concerning the second and third integrals here given analogous to that carried through in the case of the first, and to show that their mean

[6] P. J. Daniell, A General Form of Integral. Annals of Mathematics, Series 2, Vol. 19, pp. 279–294.

squares exist and have an upper bound independent of A. This independence follows at once from the fact that

$$\int_A^{A+1}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2dt=\int_0^1\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau-A)d\tau\right]^2dt \tag{26}$$

combined with the fact that

$$\int_{-\infty}^{\infty}\Phi(\alpha)d\alpha=0$$

so that

$$\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau$$

only involves the differences between values of g.

It follows that the chance that some

$$\left|\frac{1}{2a}\int_{-a}^{a}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)\,d\tau\right]^2dt-r^2\sum_1^{\infty}\frac{1}{\lambda_n}\right|>3P$$

for $a\geq N$ is less than or equal to

$$S\left(\frac{1}{N^2}+\frac{1}{(N+1)^2}+\frac{1}{(N+2)^2}+\cdots\right)\leq\frac{S}{N} \tag{27}$$

and hence can be made arbitrarily small by taking N large enough. Hence the outer chance that there is no N such that

$$\left|\frac{1}{2a}\int_{-a}^{a}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2dt-r^2\sum_1^{\infty}\frac{1}{\lambda_n}\right|\leq3P$$

for all $a\geq N$ is 0. This results from certain theorems of Daniell concerning the integrability of the limit of a monotone sequence of functionals. Since the inner chance is also 0, we see that the expression

$$\overline{\lim_{a\to\infty}}\left|\frac{1}{2a}\int_{-a}^{a}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2dt-r^2\sum_1^{\infty}\frac{1}{\lambda_n}\right|\leq3P \tag{28}$$

except for a set of cases of probability zero.

If we remember that the probability of the disjunction of a

denumerable set of events does not exceed the sum of their probabilities, we see that

$$\lim_{a\to\infty}\frac{1}{2a}\int_{-a}^{a}\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g(\tau)d\tau\right]^2dt=r^2\int_{-\infty}^{\infty}d\sigma\left[\int_{-\infty}^{\sigma}\Phi(\alpha)d\alpha\right]^2 \quad (29)$$

except for a set of cases of probability zero.

§4. **The possibility of the harmonic analysis of an irregular motion.**

It further follows from Daniell's theorems — which involve Zermelo's axiom in their proof — that the null set has measure zero, and that hence any set of functions of probability different from zero contains functions. Hence we get the following theorem: *given any denumerable set of functions* $\Phi_n(\tau)$ *satisfying conditions* (6), *the relations*

$$\lim_{a\to\infty}\frac{1}{2a}\int_{-a}^{a}\left[\int_{-\infty}^{\infty}\Phi_n(\tau-t)g(\tau)d\tau\right]^2dt=r^2\int_{-\infty}^{\infty}d\sigma\left[\int_{-\infty}^{\sigma}\Phi_n(\alpha)d\alpha\right]^2 \quad (30)$$

will all hold except in a set of cases of zero probability, and will hence hold in a non-null set of cases.

Let us consider the particular set of cases

$$\Phi_1(\tau)=\Phi(\tau);$$

$$\Phi_k(\tau)=\int_{-\infty}^{\infty}\Phi(\sigma)\Psi_{k-1}(\tau-\sigma)d\sigma; \qquad [k>1]$$

where

$$\Psi_k(\sigma)=\Psi_k(-\sigma)=\frac{2}{\pi}\int_0^{\infty}\Psi_k(u)\cos u\sigma\, du,$$

and

$$\psi_k(u)=1; \qquad [0\leq u\leq D_k]$$

$$0\leq\psi_k(u)\leq 1; \qquad [D_k\leq u\leq D_k+\epsilon_k]$$

$$\psi_k(u)=0; \qquad [D_k+\epsilon_k\leq u]$$

while all the functions $\psi_k(u)$ possess at every point derivatives of all orders. We suppose the D_k's to form an everywhere dense set, and the ϵ_k's for each D_k to assume arbitrarily small values.

We may verify at once that if conditions (6) hold for $\Phi_1(\tau)$, they hold for all Φ_k's. If we put

$$\psi(u) = \int_{-\infty}^{\infty} \Phi(\tau) e^{iu\tau} d\tau, \tag{31}$$

we get

$$\begin{aligned}\int_{-\infty}^{\infty} \Phi_k(\tau) e^{iu\tau} d\tau &= \int_{-\infty}^{\infty} e^{iu\tau} d\tau \int_{-\infty}^{\infty} \Phi(\sigma) \Psi_{k-1}(\tau-\sigma) d\sigma \\ &= \int_{-\infty}^{\infty} \Phi(\sigma) e^{iu\tau} d\sigma \int_{-\infty}^{\infty} \Psi_{k-1}(\tau-\sigma) e^{iu(\tau-\sigma)} d\tau \\ &= \psi(u)\psi_{k-1}(u).\end{aligned}$$

Hence

$$\Phi_1(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(u) e^{-iu\tau} du;$$

$$\Phi_k(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(u)\psi_{k-1}(u) e^{-iu\tau} d\,u. \qquad [k>0]$$

Upon integration, we find that

$$\int_{-\infty}^{\sigma} \Phi_1(\alpha) d\alpha = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\psi(u)}{u} e^{-iu\sigma} du$$

$$\int_{-\infty}^{\sigma} \Phi_k(\alpha) d\alpha = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\psi(u)}{u} \psi_{k-1}(u) e^{-iu\sigma} du. \qquad [k>0]$$

It is to be noted that it follows from (6) that $\psi(u) = O(u)$ at the origin. Consequently,

$$\int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\sigma} \left[\Phi_1(\alpha) d\alpha\right]^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{[\psi(u)]^2}{u^2} du;$$

$$\int_{-\infty}^{\infty} d\sigma \left[\int_{-\infty}^{\sigma} \Phi_k(\alpha) d\alpha\right]^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{[\psi(u)]^2 [\psi_{k-1}(u)]^2}{u^2} du. \tag{32}$$

We thus get

$$\lim_{D_{k-1} \to \infty} \int_{\infty}^{\infty} d\sigma \left[\int_{-\infty}^{\sigma} \Phi_k(\alpha) d\alpha\right]^2 = \int_{-\infty}^{\infty} d\sigma \left[\int_{-\infty}^{\sigma} \Phi_1(\alpha) d\alpha\right]^2. \tag{33}$$

We have now established that except in a set of cases of zero probability,

$$h(t) = \int_{-\infty}^{\infty} \Phi(\tau - t) g(\tau) d\tau$$

possesses a generalized trigonometric analysis in the sense of the paper to which this is a sequel, for we have demonstrated that conditions (*a*), (*b*), and (*c*) of the introduction to this paper are fulfilled.

§5. **The spectrum of an irregular motion.**

We showed in the previous paper that if

$$\Gamma(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} h(t) \frac{1-\cos ut}{t^2}\, dt;$$

$$\Delta(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} h(t) \frac{ut\, e^{-t^2} - \sin ut}{t^2} dt;$$

$$H_k(t) = \int_{-\infty}^{\infty} \Phi_k(\tau - t) g(\tau) d\tau;$$

we have

$$\lim_{A \to \infty} \frac{1}{2A}\int_{-A}^{A} [H_k(t)]^2 dt = \frac{3}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon^3} \int_0^{\infty} [\psi_{k-1}(u)]^2$$
$$\left\{ [\Delta(u+\epsilon) + \Delta(u-\epsilon) - 2\Delta(u)]^2 + [\Gamma(u+\epsilon) + \Gamma(u-\epsilon) - 2\Gamma(u)]^2 \right\} du. \tag{34}$$

Hence except for a set of cases of zero probability,

$$\frac{3}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon^3} \int_0^{\infty} [\psi_{k-1}(u)]^2 \left\{ [\Delta(u+\epsilon) + \Delta(u-\epsilon) - 2\Delta(u)]^2 \right.$$
$$\left. + [\Gamma(u+\epsilon) + \Gamma(u-\epsilon) - 2\Gamma(u)]^2 \right\} du$$
$$= r^2 \int_{-\infty}^{\infty} d\sigma \left[\int_{-\infty}^{\sigma} \Phi_k(\alpha) d\alpha \right]^2 = \frac{r^2}{2\pi} \int_{-\infty}^{\infty} \frac{[\psi_{k-1}(u)]^2 [\psi(u)]^2}{u^2} du. \tag{35}$$

We therefore have

$$\limsup_{D_k + \epsilon_k \leq U} \frac{3}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon^3} \int_0^{\infty} [\psi_k(u)]^2$$
$$\left\{ [\Delta(u+\epsilon) + \Delta(u-\epsilon) - 2\Delta(u)]^2 + [\Gamma(u+\epsilon) + \Gamma(u-\epsilon) - 2\Gamma(u)]^2 \right\} du$$
$$= \frac{r^2}{2\pi} \int_{-U}^{U} \frac{[\psi(u)]^2}{u^2} du.$$

In the preceding paper, we use the notation,

$$\limsup_{D_k+\epsilon_k\leq U} 3\lim_{\epsilon\to 0}\frac{1}{\epsilon^3}\int_0^\infty [\psi_k(u)]^2$$
$$\{[\Delta(u+\epsilon)+\Delta(u-\epsilon)-2\Delta(u)]^2+[\Gamma(u+\epsilon)+[(u-\epsilon)-2\Gamma(u)]^2\}du$$
$$=R\{h;U\}.$$

It follows that

$$R\{h;u\}=\frac{r^2}{\pi}\int_{-u+0}^{u-0}\frac{[\psi(u)]^2}{u^2}du. \tag{36}$$

In the preceding paper, we prove that

$$C(h;\xi)=\lim_{\epsilon\to 0}\frac{3}{2\pi\epsilon^3}\int_{-\infty}^{\infty}h\left(t-\frac{\xi}{2}\right)h\left(t+\frac{\xi}{2}\right)\frac{\sin^4\epsilon t}{t^4}dt$$
$$=\frac{1}{2}\int_0^\infty\cos\alpha\xi\,dR\{h;\alpha\}.$$

This becomes

$$C(h;\xi)=\frac{r^2}{2\pi}\int_{-\infty}^{\infty}\cos\alpha\,\xi\frac{[\psi(\alpha)]^2}{\alpha^2}d\alpha$$
$$=\frac{r^2}{2\pi}\int_{-\infty}^{\infty}\left[\cos^2\frac{\alpha\xi}{2}-\sin^2\frac{\alpha\xi}{2}\right]\frac{[\psi(\alpha)]^2}{\alpha^2}d\alpha$$
$$=\frac{r^2}{2\pi}\int_0^\infty\left[\cos^2\frac{\alpha\xi}{2}-\sin^2\frac{\alpha\xi}{2}\right]$$
$$\frac{[\psi(\alpha)+\psi(-\alpha)]^2+[\psi(\alpha)-\psi(-\alpha)]^2}{\alpha^2}d\alpha$$
$$=\frac{r^2}{4}\int_0^\infty\left\{\left[\int_0^\infty\cos\frac{\alpha\xi}{2}\frac{\psi(\alpha)+\psi(-\alpha)}{2\alpha}\cos\alpha x\,d\alpha\right]^2\right.$$
$$-\left[\int_0^\infty\sin\frac{\alpha\xi}{2}\frac{\psi(\alpha)+\psi(-\alpha)}{2\alpha}\sin\alpha x\,d\alpha\right]^2$$
$$+\left[\int_0^\infty\cos\frac{\alpha\xi}{2}\frac{\psi(\alpha)-\psi(-\alpha)}{2\alpha}\sin\alpha x\,d\alpha\right]^2$$
$$\left.-\left[\int^\infty\sin\frac{\alpha\xi}{2}\frac{\psi(\alpha)-\psi(-\alpha)}{2\alpha}\cos\alpha x\,d\alpha\right]^2\right\}dx$$

$$= \frac{r^2}{4}\int_0^\infty \left\{ \left[\int_{-\infty}^{\infty} \frac{\cos a\left(x+\frac{\xi}{2}\right)+\cos a\left(x-\frac{\xi}{2}\right)}{2a}\psi(a)da\right]^2 \right.$$

$$- \left[\int_{-\infty}^{\infty} \frac{\cos a\left(x+\frac{\xi}{2}\right)-\cos a\left(x-\frac{\xi}{2}\right)}{2a}\psi(a)da\right]^2$$

$$+ \left[\int_{-\infty}^{\infty} \frac{\sin a\left(x+\frac{\xi}{2}\right)+\sin a\left(x-\frac{\xi}{2}\right)}{2a}\psi(a)da\right]^2$$

$$\left. - \left[\int_{-\infty}^{\infty} \frac{\sin a\left(x+\frac{\xi}{2}\right)-\sin a\left(x-\frac{\xi}{2}\right)}{2a}\psi(a)da\right]^2 \right\} dx$$

$$= \frac{r^2}{4}\int_0^\infty dx \int_{-\infty}^{\infty} \frac{\psi(a)}{a}\cos a\left(x+\frac{\xi}{2}\right)da \int_{-\infty}^{\infty} \frac{\psi(\beta)}{\beta}\cos\beta\left(x-\frac{\xi}{2}\right)d\beta$$

$$+ \frac{r^2}{4}\int_0^\infty dx \int_{-\infty}^{\infty} \frac{\psi(a)}{a}\sin a\left(x+\frac{\xi}{2}\right)da \int_{-\infty}^{\infty} \frac{\psi(\beta)}{\beta}\sin\beta\left(x-\frac{\xi}{2}\right)d\beta$$

$$= \frac{r^2}{4}\int_0^\infty dx \int_{-\infty}^{\infty} \frac{\psi(a)}{a}da \int_{-\infty}^{\infty} \frac{\psi(\beta)}{\beta}d\beta \cos\left[(a+\beta)x+(a-\beta)\frac{\xi}{2}\right]$$

$$= \frac{r^2}{8}\int_0^\infty dx \int_{-\infty}^{\infty} \frac{\psi(a)}{a}da$$

$$\int_{-\infty}^{\infty} \frac{\psi(\beta)}{\beta}d\beta \left[e^{i\left[(a+\beta)x+(a-\beta)\frac{\xi}{2}\right]} + e^{-i\left[(a+\beta)x+(a-\beta)\frac{\xi}{2}\right]}\right]. \quad (37)$$

We now refer back to formula (31), which when inverted yields us

$$\Phi(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(u)e^{-iu\tau}du.$$

By integrating both sides of this expression, we obtain

$$\int_{-\infty}^{\tau} \Phi(\tau)d\tau = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\psi(u)}{u}e^{-iu\tau}\,du. \quad (38)$$

If we substitute this result into (37), we get

$$C(h;\xi)=\frac{\pi^2}{2}r^2\int_0^\infty dx\left[\int_{-\infty}^{-x-\frac{\xi}{2}}\Phi(\tau_1)d\tau_1\int_{-\infty}^{-x+\frac{\xi}{2}}\Phi(\tau_2)d\tau_2\right.$$

$$\left.+\int_{-\infty}^{x+\frac{\xi}{2}}\Phi(\tau_3)d\tau_3\int_{-\infty}^{x-\frac{\xi}{2}}\Phi(\tau_4)d\tau_4\right]$$

$$=\frac{\pi^2}{2}r^2\int_{-\infty}^{\infty}dx\int_{-\infty}^{x+\frac{\xi}{2}}\Phi(\tau_1)d\tau_1\int_{-\infty}^{x-\frac{\xi}{2}}\Phi(\tau_2)d\tau_2\,. \tag{39}$$

That is, the mean of $h\left(t-\frac{\xi}{2}\right)h\left(t+\frac{\xi}{2}\right)$ differs only by a constant factor from the integral of $\int_{-\infty}^{x+\frac{\xi}{2}}\Phi(\tau_1)d\tau_1\int_{-\infty}^{x-\frac{\xi}{2}}\Phi(\tau_2)d\tau_2$.

We may regard $R\{h;u\}$ as a measure of the sum of the squares of the coefficients of the terms in the trigonometric expansion of $h(t)$ up to the term corresponding to frequency $u/2\pi$. We may regard $h(t)$ as the velocity of a motion in which the displacement is

$$\eta(t)=\int_{-\infty}^{\infty}g(\tau)d\tau\int_{-\infty}^{\tau-t}\Phi(\sigma)d\sigma.$$

We then see that $R\{h;u\}$ is proportional to the total power in the spectrum of this motion, up to the term of frequency $u/2\pi$. By (32), this only differs by a constant factor from the integral of the square of the amplitudes in the spectrum of $\int_{-\infty}^{\tau}\Phi(\sigma)d\sigma$ over the same range of frequencies. We may put this more expressively if we say that the power-spectrum of a haphazard succession of impulses of a given form is the same as the energy-spectrum of one of these impulses, or again if we say that the power-spectrum of the response of a linearly resonant system to a succession of haphazard sharp impulses is the same as the energy-spectrum of the response of the system to a single such impulse. The enunciation of this as a physical fact is not new, but the rigor-

ous mathematical proof is new, to the best of the author's knowledge and belief. A full discussion is to be found in the next section.

While the function $g(t)$ has no power-spectrum, in the sense here defined, we may regard a motion with displacement $x=g(t)$ as having the distribution of power in its spectrum corresponding to the limit of the distribution of power in the spectrum of a motion with displacement

$$G(t)=\int_{-\infty}^{\infty} g(\tau)\Psi_k(t-\tau)d\tau$$

or with velocity

$$G'(t)=\int_{-\infty}^{\infty} g(\tau)\Psi_k'(t-\tau)d\tau.$$

We have, however,

$$R\{G';u\}=\frac{r^2}{\pi}\int_{-u}^{u}[\psi_k(u)]^2du=\frac{2ur^2}{\pi} \qquad [D_k\geq u]$$

and

$$\lim_{D_k\to\infty} R\{G';u\}=\frac{2ur^2}{\pi}. \tag{40}$$

That is, in a generalized sense, the power-spectrum of the function $g(t)$ may be regarded as uniformly distributed in frequency, in all except an assemblage of cases of zero probability.

§6. The Schroteffekt.

A linear system such as a resonator may be regarded as generating a linear transformation of one function, representing the impressed displacement, into another function, representing the emergent displacement. For example, the horn of the phonograph generates a transformation of the displacement of the diaphragm into the displacement of the air at a specified place. Precisely similar cases occur in electrical systems.

In all linear systems occurring in practice, there are frictional losses of energy. If these did not exist, the response of the system to impressed periodic displacements of certain frequencies would

tend to infinity with increasing time. Such unlimited resonance is however never found.

The linear operations generated by resonators are said to be operators of the closed cycle. This means that the result of applying one of these operators to a given function is not affected by a displacement of the origin in time. In symbols, if I is such an operator,

$$I\{f(t+\tau)\} = [I\{f(\sigma)\}]_{\sigma=t+\tau}. \tag{41}$$

All resonators found in practice respond only slightly to very high frequencies. Such resonators have characteristic operators of the form

$$I\{f(t)\} = \int_{-\infty}^{\infty} F(\tau - t) f(\tau) d\tau.$$

If such a resonator is subjected to a displacement $g(t)$, the velocity corresponding to the emergent displacement is

$$h(t) = -\int_{-\infty}^{\infty} F'(\tau - t) g(\tau) d\tau,$$

provided F and F' are summable and of summable square over the whole line. If in addition $F'(\tau)$ is $O(-\tau^2)$ at $\pm\infty$, condition (6) is fulfilled for F', and we have almost always

$$R\{h; u\} = \frac{r^2}{\pi} \int_{-u}^{u} \left[\int_{-\infty}^{\infty} F(\tau) e^{i\omega\tau} d\tau \right]^2 du, \tag{42}$$

provided $g(t)$ has a normal distribution.

As a particular application: let us consider the response of an acoustical resonator to a noise consisting of a series of sharp impulses to the air, such as those produced by the grains of sand in a sand-blast, which are (1) short in duration as compared with the period of an audible sound, (2) so frequent that many occur within the period of an audible sound, (3) occurring at random intervals, and (4) such that the quantities determining the character of this random distribution do not vary with the time. If we make a harmonic analysis of the output of the resonator, in

the sense of the paper to which this is a sequel, we shall find that the value of the R of this function for a given frequency is proportional to the energy of the response of the oscillator to a sinusoidal disturbance of this frequency and of unit energy. This property may be employed in the calibration of phonodeiks, in which the output function is permanently recorded as a curve, which may be analyzed by the methods we have given.

Another example is formed by the so-called Schroteffekt of Schottky.[7] The space current in a vacuum tube is carried by electrons, and hence can not be continuous. Its deviations from steadiness are of such a nature as to make the quantity of electricity passing in a given time a function with the same sort of distribution as $g(t)$. If the tube is connected to a linear system of inductances, capacities, and resistances, the irregularities of the space current will produce oscillations of the system. These can be amplified to the point where they are audible when rectified, and have been observed. The question is as to the distribution in frequency of the electric power in some member of the oscillating system. Our results show that it is proportional to the power in the same member of the system when the space current is replaced by a sinusoidal current of the given frequency and of unit power. We shall not here discuss the factor of proportionality in question, which has been adequately treated in the papers referred to, and which gives a valuable means of measuring the electronic charge.

The observation that the space current is uniformly distributed with respect to power over the different frequencies was made by Schottky, although the means of rigorously proving this were not at hand. Fry's paper is in large measure a polemic against the idea of making a harmonic analysis of an irregular motion. He points out that the notion of the amplitude of a component of the spectrum of the electron stream does not generalize properly when the number of electrons is taken to be very large, and draws from that the incorrect conclusion that the notion of power — he says " energy " — will not generalize.

[7] Cf. W. Schottky, Ann. der Phys., 57, 541–67 (1918); 68, 157–76 (1922); T. C. Fry, Jour. Frank. Inst., 199, 203–221 (1925).

§7. White light and interference.

It was first pointed out by Gouy[8] and later emphasized by Schuster[9] that white light — that is, light with a uniform power distribution over its spectrum — may be regarded as being made up of a succession of perfectly irregular impulses. This is completely confirmed by the results of our last section. The author wishes, however, to take this opportunity to point out that neither Gouy nor Schuster have given a clear and unambiguous definition of what is meant by " a succession of perfectly irregular impulses," and that to the best of the author's knowledge, no one has yet exhibited a function possessing a continuous spectrum in the sense of a spectrum with a continuous power distribution over the range of frequencies. The Fourier series and the analogous series for Bohr's " almost periodic functions " represent nothing but line spectra. The continuous spectra of the Fourier integral have a continuous distribution of energy over the spectrum, but represent processes of zero power. They are the spectra of pulses, of transient phenomena, whereas it is highly desirable to have a theory of light with a continuous spectrum as a steady-state phenomenon. The present paper, therefore, forms a necessary part of the existence theory of physical optics.

The author wishes to point out its application to the theory of coherent and incoherent sources. Let $g_1(t)$ and $g_2(t)$ be two independently varying functions, each with a normal distribution of the type discussed in this paper. Consider

$$\left[\int_{-\infty}^{\infty}\Phi(\tau-t)[g_1(\tau)+g_2(\tau)]d\tau\right]^2$$
$$=\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g_1(\tau)d\tau\right]^2+\left[\int_{-\infty}^{\infty}\Phi(\tau-t)g_2(\tau)d\tau\right]^2$$
$$+2\int_{-\infty}^{\infty}\Phi(\tau-t)g_1(\tau)d\tau\int_{-\infty}^{\infty}\Phi(\tau-t)g_2(\tau)d\tau. \qquad (43)$$

The mean of the last term of this with respect to the time is

$$\frac{1}{A}\int_{-A}^{A}dt\int_{-\infty}^{\infty}\Phi(\tau-t)g_1(\tau)d\tau\int_{-\infty}^{\infty}\Phi(\tau-t)g_2(\tau)d\tau. \qquad (44)$$

If we make use of our theorems on the average of analytic function-

[8] Journal de Physique, Vol. V., p. 354 (1886).
[9] Schuster and Nicholson, Theory of Optics (Third Edition), p. 333.

als, we see that the mean square value of this, taken separately with respect to g_1 and g_2, is

$$\frac{\tau^2}{A^2}\int_{-A}^{A} ds \int_{-A}^{A} dt \left[\int_0^\infty \Phi(\sigma-s)\,d\sigma \int^\sigma \Phi(\tau-t)\tau\,d\tau - \int_\infty^0 \Phi(\sigma-s)\,d\sigma \int_\sigma^0 \Phi(\tau-t)\tau\,d\tau\right]^2 = O(1/A^2). \tag{45}$$

An argument such as that which we have developed in §3 will now show that in all except a set of cases of zero probability,

$$\lim_{A\to\infty} \frac{1}{A}\int_{-A}^{A} dt \int_{-\infty}^{\infty} \Phi(\tau-t)g_1(\tau)\,d\tau \int_{-\infty}^{\infty} \Phi(\tau-t)g_2(\tau)\,d\tau = 0.$$

It follows that in all except a set of cases of zero probability,

$$M\left[\int_{-\infty}^{\infty} \Phi(\tau-t)[g_1(\tau)+g_2(\tau)]\,d\tau\right]^2 = M\left[\int_{-\infty}^{\infty} \Phi(\tau-t)g_1(\tau)\,d\tau\right]^2 + M\left[\int_{-\infty}^{\infty} \Phi(\tau-t)g_2(\tau)\,d\tau\right]^2. \tag{46}$$

That is, if we have two independent sources of light, even though they may have identical power spectra, it is almost certain that their energies will be additive, and that they cannot interfere. On the other hand, we have

$$M\left[\int_{-\infty}^{\infty} \Phi(\tau-t)[g(\tau)+g(\tau)]\,d\tau\right]^2 = 4M\left[\int_{-\infty}^{\infty} \Phi(\tau-t)g(\tau)\,d\tau\right]^2;$$
$$M\left[\int_{-\infty}^{\infty} \Phi(\tau-t)[g(\tau)-g(\tau)]\,d\tau\right]^2 = 0; \tag{47}$$

thus showing the possibility of interference between light from non-independent sources.

We shall say that two functions $h_1(t)$ and $h_2(t)$ are *coherent* at frequency $u/2\pi$ if there exist quantities α and $\rho(\alpha)$ such that

$$0 \not\equiv \rho(\alpha) = \left| \lim_{\epsilon\to 0} \frac{\left\{\begin{array}{l} R\{h_1(t+\alpha)+h_2(t),\ u+\epsilon\} \\ \quad -R\{h_1(t+\alpha)+h_2(t),\ u-\epsilon\} \\ -R\{h_1(t+\alpha)-h_2(t),\ u+\epsilon\} \\ \quad +R\{h_1(t+\alpha)-h_2(t),\ u-\epsilon\} \end{array}\right\}}{4\{R\{h_1(t+\alpha),\ u+\epsilon\} - R\{h_1(t+\alpha),\ u-\epsilon\}\}^{1/2} \cdot \{R\{h_2(t),\ u+\epsilon\} - R\{h_2(t),\ u-\epsilon\}\}^{1/2}} \right| \tag{48}$$

$\max_a \rho(a)$ will then be said to be the measure of their coherence, and the corresponding angle a their phase angle. If we consider two components of the electric vector of light at right angles to one another and to the direction of its propagation, and represent them by $h_1(t)$ and $h_2(t)$, respectively, we shall call $100 \max_a \rho(a)$ the *percentage of polarization* of the light at frequency u. If $\max_a \rho(a) = 1$, the directions making with the direction of oscillation of $h_1(t)$ angles θ satisfying the conditions

$$\sin 2\theta = \frac{-l_1^2 \sin 2u\,a}{l_1^2 + l_2^2 - 2l_1 l_2 \cos 2\,ua}, \tag{49}$$

where

$$l_1^2 = \lim_{\epsilon \to 0} \frac{R\{h_1,\, u+\epsilon\} - R\{h_1,\, u-\epsilon\}}{2\epsilon};$$

$$l_2^2 = \lim_{\epsilon \to 0} \frac{R\{h_2,\, u+\epsilon\} - R\{h_2,\, u-\epsilon\}}{2\epsilon}; \tag{50}$$

are the directions of the axes of the elliptical polarization of our light, and the quantities

$$l_1 \sin\theta + l_2 \cos\theta;$$
$$l_1 \cos\theta - l_2 \sin\theta; \tag{51}$$

are these axes themselves. It is, of course, understood that all these are taken with respect to light of frequency $u/2\pi$. We shall not go into the question of the existence of the limits in question under general hypotheses, but it is a simple matter to produce cases where they exist.

§8. Taylor's theory of irregular motion.

In a paper entitled, "Diffusion by Continuous Movement,"[10] G. I. Taylor developed a theory of irregular motion depending on the idea of the correlation between a function and itself under a displacement, which we have borrowed from him in the present paper. He also introduces the notion of the coefficient of correlation between two derivatives of a function. His paper does not aim at a high mathematical rigor, and the precise conditions for the validity of his theorems are not stated with all desirable defi-

[10] G. I. Taylor, Proc. London Math. Soc., Ser. 2, 20, pp. 196-212 (1920).

niteness. He appears to confine himself, however, to analytic functions.

It will be the purpose of the present section to prove his results under fairly general hypotheses. We shall then discuss the harmonic analysis of the various derivatives of the functions of our last paper. We have.

$$\frac{1}{h^3}\int_{-\infty}^{\infty} f(t)f'(t)\frac{\sin^4 ht}{t^4}\,dt$$
$$= \left\{\frac{1}{h^3}[f(t)]^2\frac{\sin^4 ht}{t^4}\right\}_{-\infty}^{\infty} - \frac{1}{h^3}\int_{-\infty}^{\infty} f'(t)f(t)\frac{\sin^4 ht}{t^4}\,dt$$
$$-\frac{1}{h^3}\int_{-\infty}^{\infty}[f(t)]^2\frac{d}{dt}\left(\frac{\sin^4 ht}{t^4}\right)dt.$$

Consequently

$$\frac{1}{h^3}\int_{-\infty}^{\infty} f(t)f'(t)\,\frac{\sin^4 ht}{t^4}\,dt = -\frac{1}{2h^3}\int_{-\infty}^{\infty}[f(t)]^2\frac{d}{dt}\left(\frac{\sin^4 ht}{t^4}\right)dt$$
$$= -\frac{h}{2}\int_{-\infty}^{\infty}[f(u/h)]^2\frac{d}{du}\frac{\sin^4 u}{u^4}\,du. \qquad (52)$$

Hence

$$\left|\frac{1}{h^3}\int_{-\infty}^{\infty} f(t)f'(t)\,\frac{\sin^4 ht}{t^4}\,dt\right| \leq 2h\,\max\left|\frac{u\sin u\cos u-\sin^2 u}{u^3}\right|$$
$$\cdot\int_{-\infty}^{\infty}[f(u/h)]^2\frac{\sin^2 u}{u^2}\,du. \qquad (53)$$

It follows that if

$$\frac{1}{h}\int_{-\infty}^{\infty}[f(u)]^2\frac{\sin^2 hu}{u^2}\,du$$

is bounded,

$$\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty} f(t)f'(t)\,\frac{\sin^4 ht}{t^4}\,dt = 0. \qquad (54)$$

In general, if

$$\frac{1}{h}\int_{-\infty}^{\infty}[f^{(k)}(u)]^2\frac{\sin^2 hu}{u^2}\,du \qquad [k=1, 2, \ldots, n]$$

exists, we have

$$\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}[f(t)f^{(n)}(t)+f^{(k)}(t)+f^{(n-k)}(t)\,(-1)^k]\frac{\sin^4 ht}{t^4}\,dt = 0. \qquad (55)$$

That is, the mean value of $f^{(k)}(t)f^{(n-k)}(t)$ is 0 if n is odd, the average of $[f^{(n/2)}(t)]^2$ if $k-n/2$ is divisible by 2, and the negative average of $[f^{(n/2)}(t)]^2$ otherwise. This is essentially the theorem of G. I. Taylor. Taylor then goes on to formulate the Maclaurin expansion of what is essentially

$$\lim_{h\to 0} \frac{3}{2\pi h^3}\int_{-\infty}^{\infty} f(t)f(t+\xi)\frac{\sin^4 ht}{t^4}dt$$

in powers of ξ. This cannot be justified without much more stringent restrictions on f, and is not employed by him in his further work.

It is a matter of a certain amount of interest to ascertain a set of conditions under which $f(t)$ and $f'(t)$ can be simultaneously given a harmonic analysis. For this we shall need to employ certain auxiliary functions. Let $\Phi^{(1)}(t)$ be an odd summable function of summable square which is $O(1/t^2)$ at infinity. Let $f(t)$ be a function which fulfills the hypotheses necessary for a harmonic analysis in the sense of our previous paper. Let us consider the function

$$g_1(t) = \int_{-\infty}^{\infty} \Phi^{(1)}(\tau - t)f(\tau)d\tau.$$

In our previous paper we showed that if $\Phi^{(2)}(t)$ was an even function, summable and of summable square, which is $O(1/t^2)$ at infinity, and if

$$g_2(t) = \int_{-\infty}^{\infty} \Phi^{(2)}(\tau - t)f(\tau)d\tau$$

then g_2 is harmonically analyzable and

$$R\{g_2; u\} = \int_0^{u-0}\left[\int_{-\infty}^{\infty}\Phi^{(2)}(\xi)\cos u\,\xi\, d\xi\right]^2 dR\{f; u\}. \qquad (56)$$

Precisely similar methods show that g_1 is harmonically analyzable and that

$$R\{g_1; u\} = \int_0^{u-0}\left[\int_{-\infty}^{\infty}\Phi^{(1)}(\xi)\sin u\,\xi d\xi\right]^2 dR\{f; u\}. \qquad (57)$$

Now let $M\{[f(t)]^2\}$ and $M\{[f'(t)]^2\}$ exist, and let

$$\lim_{h\to 0}\frac{1}{h^2}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(\tau)\Psi_h(t-\tau)d\tau\right]^2\frac{\sin^4 ht}{t^4}dt$$

exist for an everywhere dense set of D_h's and for arbitrarily small ϵ's. The arguments of the previous paper will show that

$$\frac{3}{\pi}\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f'(\tau)\Psi_h(t-\tau)\,d\tau\right]^2\frac{\sin^4 ht}{t^4}\,dt$$

$$=\frac{3}{\pi}\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(\tau)\Psi_h'(t-\tau)d\tau\right]^2\frac{\sin^4 ht}{t^2}dt$$

will exist, and will in fact equal

$$\int^{\infty}\frac{[\Psi_h(u)]^2}{u^2}\,dR\{f;u\} \tag{58}$$

Let us make the additional assumption that

$$M\{[f'(t)]^2\}$$

$$=\lim_{D_h\to\infty}\frac{3}{2\pi}\lim_{h\to 0}\frac{1}{h^3}\int_{\infty}^{\infty}\left[\int_{-\infty}^{\infty}f'(\tau)\Psi_h(t-\tau)d\tau\right]^2\frac{\sin^4 ht}{t^4}\,dt. \tag{59}$$

The function $f'(t)$ will then have a harmonic analysis in the sense of the previous paper. Now over any finite range

$$f(t)-\int_{-\infty}^{\infty}f(\tau)\Psi_h(t-\tau)d\tau=\operatorname*{l.\,m.}_{D_l\to\infty}\int_{-\infty}^{\infty}f(\sigma)[\Psi_h(t-\sigma)-\Psi_h(t-\sigma)]d\sigma$$

$$=\operatorname*{l.\,m.}_{D_l\to\infty}\int_{-\infty}^{\infty}f'(\tau)d\tau\int_{-\infty}^{\tau}[\Psi_l(t-\sigma)-\Psi_h(t-\sigma)]d\sigma$$

$$=\operatorname*{l.\,m.}_{D_l\to\infty}\frac{2}{\pi}\int_{-\infty}^{\infty}f'(\tau)d\tau\int_0^{\infty}\cos u(t-\tau)\frac{\psi_l(u)-\psi_h(u)}{u}\,du$$

$$=\frac{2}{\pi}\int_{-\infty}^{\infty}f'(\tau)d\tau\int_0^{\infty}\cos u(t-\tau)\,\frac{1-\psi_h(u)}{u}\,du \tag{60}$$

except possibly over a set of zero measure. The function

$$\frac{2}{\pi}\int_0^{\infty}\cos u\tau\,\frac{1-\psi_h(u)}{u}du$$

is summable and of summable square, and is $O(1/\tau^3)$ at infinity. Hence

$$\frac{3}{\pi}\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}\left[f(t)-\int_{-\infty}^{\infty}f(\tau)\Psi_h(t-\tau)d\tau\right]^2\frac{\sin^4 ht}{t^4}\,dt$$

exists, and equals

$$\int_{-\infty}^{\infty}\frac{[1-\psi_h(u)]^2}{u^2}dR\{f;u\}. \tag{61}$$

It follows that

$$\lim_{D_h\to\infty}\frac{3}{\pi}\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}\left[f(t)-\int_{-\infty}^{\infty}f(\tau)\Psi_h(t-\tau)d\tau\right]^2\frac{\sin^4 ht}{t^4}\,dt=0$$

and by the application of a form of the Schwarz inequality,

$$M\{[f(t)]^2\}=\lim_{D_h\to\infty}\frac{3}{2\pi}\lim_{h\to 0}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(\tau)\Psi_h(t-\tau)d\tau\right]^2\frac{\sin^4 ht}{t^4}\,dt. \tag{62}$$

That is, $f(t)$ also has a harmonic analysis in the sense of the previous paper. In general, if $M\{[f(t)]^2\}$ and $M\{[f^n(t)]^2\}$ exist, and every

$$\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(\tau)\Psi_h(t-\tau)d\tau\right]^2\frac{\sin^4 ht}{t^4}\,dt$$

then

$$\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f^{(j)}(\tau)\Psi_h(t-\tau)d\tau\right]^2\frac{\sin^4 ht}{t^4}dt \quad [j=1,2,\ldots \tag{63}$$

will exist. If in addition $M\{[f^{(j)}(t)]^2\}$ $[j=1,2,\ldots,n]$ exist and

$$M\{[f^{(n)}(t)]^2\}=\lim_{D_h\to\infty}\frac{3}{2\pi}\lim_{h\to 0}\frac{1}{h^3}\int_{-\infty}^{\infty}[f^{(n)}(\tau)\Psi_h(t-\tau)d\tau]^2\frac{\sin^4 ht}{t^4}\,dt, \tag{64}$$

then every $f^{(j)}(t)$ $[j=1,\ldots,n]$ will have a harmonic analysis in the sense of the previous paper, and indeed

$$R\{f^{(j)};u\}=\int_0^{u-0}\frac{1}{u^{2j}}dR\{f;u\}. \tag{65}$$

GRASSMANNIAN GEOMETRY IN RIEMANNIAN SPACE

By C. L. E. Moore

1. **Introduction.** The space with which the present note is concerned is that of Riemann in which the element of arc is defined by

$$ds^2 = a_{rs}dx^r dx^s, \qquad r, s = 1, 2, \ldots n \tag{1}$$

(The Einstein convention of indicating summation by repeated indices will be used throughout.) A covariant vector is defined as any set of the first order v_r which transforms according to the covariant law and a contravariant vector as a set u^r which transforms according to the contravariant law. (The usual convention of using lower indices to indicate covariant quantities and upper indices to indicate contravariant quantities is strictly adhered to.) We restrict ourselves to vectors at a point in the given space. If u_r, u^s are the covariant and contravariant representations of the same vector then,

$$u_r = a_{rs}u^s, \qquad u^s = a^{rs}u_r.$$

Besides the geometric quantities determined by vectors we shall make use of three sets which we shall not so consider, *viz.*, (*a*) the sets a_{rs}, a^{rs}, (*b*) the sets $\epsilon_{r_1 \ldots r_n}$, $\epsilon^{r_1 \cdots r_n}$ used by Ricci and Levi-Civita and having the following properties:

$$\epsilon_{r_1 \ldots r_n} = \begin{cases} \sqrt{a} \text{ if } r_1 \ldots r_n \text{ is an even permutation of } 1, 2 \ldots n. \\ -\sqrt{a} \text{ if } r_1 \ldots r_n \text{ is an odd permutation of } 1, 2 \ldots n. \\ 0 \text{ if } r_1 \ldots r_n \text{ are not all distincts.} \end{cases}$$

$$\epsilon^{r_1 \ldots r_n} \quad \begin{cases} \dfrac{1}{\sqrt{a}} \text{ if } r_1 \ldots r_n \text{ is an even permutation.} \\ \dfrac{-1}{\sqrt{a}} \text{ if } r_1 \ldots r_n \text{ is an odd permutation.} \\ 0 \quad \text{if } r_1 \ldots r_n \text{ are not all distinct.} \end{cases}$$

where $a = |a_{sr}|$

(*c*) The generalized Kronecker symbol introduced by Murnaghan[1] and defined as follows

$$\delta^{s_1 \dots s_m}_{r_1 \dots r_m} = 0$$

if $r_1 \dots r_m$ and $s_1 \dots s_m$ are not arrangements of the same set of m distinct integers. If $r_1 \dots r_m$, $s_1 \dots s_m$ are arrangements of the same set of m distinct integers, then

$$\delta^{s_1 \dots s_m}_{r_1 \dots r_m} = \pm 1$$

according as it takes an even or odd number of changes to make the two arrangements alike.

So far as Riemannian space is concerned the symbols ϵ and δ are not distinct as was shown by Lipka,[2] but I have given them separately because δ can be defined even in case there is no element of arc.

The following useful formulae were given by Lipka and Murnaghan:

$$\epsilon^{i_1, i_2 \dots i_n} \epsilon_{i_1, i_2 \dots i_n} = n!,$$

$$\epsilon^{i_1, i_2 \dots i_n} \epsilon_{s_1 s_2 \dots s_m i_{m+1} \dots i_n} = (n-m)! \, \delta^{i_1 i_2 \dots i_m}_{s_1 s_2 \dots s_m},$$

$$\delta^{r_1 \dots r_m}_{s_1 \dots s_m} = \frac{1}{m!} \delta^{r_1 \dots r_m}_{a_1 \dots a_m} \delta^{a_1 \dots a_m}_{s_1 \dots s_m}. \tag{3}$$

2. **Progressive product.** Murnaghan showed that the progressive product of m vectors (*i.e.*, the space determined by them) is

$$U_{s_1 \dots s_m} = \delta^{r_1 \dots r_m}_{s_1 \dots s_m} u_{1/r_1} u_{2/r_2} \dots u_{m/r_m}.$$

If $U_{s_1 \dots s_m}$, $V_{t_1 \dots t_p}$ are two spaces determined by m and p vectors respectively and if $m+p<n$, then the progressive product of U and V is obviously

$$\delta^{s_1 \dots s_m t_1 \dots t_p}_{r_1, r_2 \dots r_{m+p}} U_{s_1 \dots s_m} V_{t_1 \dots t_p} = W_{r_1 \dots r_{m+p}}.$$

[1] Bulletin Amer. Math. Soc., Vol. 31, page 323, 1925.
[2] Sui systeme E ecc. Rend. Lincei, series 5, Vol. 31.

By use of (3) it is easily seen that

$$W_{r_1 \dots r_m} = \delta^{s_1 \dots s_m t_1 \dots t_p}_{r_1 \dots r_m} u_{1/s_1} u_{2/s_2} \dots u_{m/s_m} v_{1/t_1} v_{2/t_2} \dots v_{p/t_p}$$

where $$U_{s_1 \dots s_m} = \delta^{t_1 \dots t_m}_{s_1 s_2 \dots s_m} u_{1/t_1} \dots u_{m/t_m}$$

and $$V_{t_1 \dots t_p} = \delta^{s_1 \dots s_p}_{t_1 \dots t_p} v_{1/s_1} \dots v_{p/s_p}.$$

3. **Complement.** It is well known that two vectors are perpendicular when and only when

$$u^r v_r = 0$$

from which it is evident that if V_{rs} are the coördinates of the space determined by the two vectors u and v, and if w is perpendicular to each of these, then

$$V_{rs} w^r = 0$$

and vice versa. The generalization to spaces of more than two dimensions is obvious. The system U,

$$U^{s_1 s_2 \dots s_{n-m}} = \epsilon^{r_1 \dots r_m s_1 \dots s_{n-m}} u_{1/r_1} u_{2/r_2} \dots u_{m/r_m} \tag{4}$$

are the contravariant cöordinates of the space completely perpendicular to the space determined by the vectors u_i. For

$$U^{s_1 \dots s_{n-m}} u_{i/r_i} = 0$$

if u_i is one of the original m vectors, since the indices of ϵ will then not be distinct. Hence the space determined by u_i is completely perpendicular to $U^{s_1 \dots s_{n-m}}$. The space $U^{s_1 \dots s_{n-m}}$ is called the *complement* of $U_{r_1 \dots r_m} = \delta^{s_1 \dots s_m}_{r_1 \dots r_m} u_{1/s_1} u_{2/s_2} \dots u_{m/s_m}$

The operation of multiplying by ϵ is equivalent to multiplication by the pseudo scalar.[3]

The complement of the space determined by

$$U_{r_1 \dots r_p}, \ V_{s_1 \dots s_q}$$

is $$\epsilon^{r_1 \dots r_p s_1 \dots s_q t_1 \dots t_{n-p-q}} U_{r_1 \dots r_p} V_{s_1 \dots s_q}$$

if $p+q<n$. The complement of contravariant spaces can be expressed in a similar manner.

[3] Wilson and Lewis, Proc. Amer. Acad., Vol. 48.

4. **Inner product.** The system

$$W^{s_1 \cdots s_q} = U_{r_1 \cdots r_p} V^{r_1 \cdots r_p s_1 \cdots s_q}$$

is the space in V completely perpendicular to U. For

$$\underline{W}^{s_1 \cdots s_q} = \epsilon^{r_1 \cdots r_p t_1 \cdots t_{n-p-q} s_1 \cdots s_q} U_{r_1 \cdots r_p} \overline{V}_{t_1 \cdots t_{n-p-q}}$$

where $\overline{V}$ is the complement of V. Hence W is the complement of the space determined by U and $\overline{V}$ and is therefore completely perpendicular to U and $\overline{V}$ and must therefore be completely perpendicular to U and lie in V. This is then the inner product defined by G. N. Lewis.[4]

The length of a vector v_r is defined to be

$$L_v^2 = v_r v^r$$

and similarly we define the magnitude of any geometric quantity as

$$M_u^2 = U_{r_1 \cdots r_p} U^{r_1 \cdots r_p}.$$

The angle between two vectors is defined as

$$\cos \theta = \frac{u_r v^r}{\sqrt{u_r u^r \cdot v^s v_s}}.$$

Two quantities of the same dimensions have a similar invariant which can be used to define the angle. Thus

$$\cos \psi = \frac{U_{r_1 \cdots r_p} V^{r_1 \cdots r_p}}{\sqrt{M_u^2 \cdot M_v^2}}.$$

The cosine of the angle between two spaces can be defined in other ways.[5] The angle between a vector v and a space $U_{r_1 \cdots r_p}$ is

$$\cos \phi = \frac{v^r \cdot p_r}{\sqrt{L_v^2 \cdot L_v^2}} = \frac{v_r \cdot p^r}{\sqrt{L_p^2 \cdot L_p^2}}$$

[4] Proc. Amer. Acad., Vol. 47.
See also Struik. Mehr dimensionalen Geometrie, Springer. Berlin, page 24.
[5] cf Segre. Encyc. d. Math. Wissenschaften, Band III$_3$, Heft 7. Mehrdimensionale Räume.

where p_r is the projection of v^r on U

$$p^r = v^{r_1} U_{r_1 \dots r_p} \quad U^{r\, r_1 \dots r_p}$$

and the angle between spaces of different dimensions can be defined in a similar manner.

5. **Regressive product.** If $p+q>n$ then the two spaces $U_{r_1 \dots r_p}$, $V_{s_1 \dots s_q}$ must intersect and the progressive product will vanish. We can, however, express the intersection in terms of the two spaces U and V. This is accomplished thus

$$(\overline{U}V) = (U\overline{V}) = W_{r_{n-p+1} \dots r_q} = \overline{U}^{r_1 \dots r_{n-p}} V_{r_1 \dots r_q}$$
$$= U_{r_1 \dots r_p} \overline{V}^{r_1 \dots r_{n-q}}$$

where $\overline{U}$, $\overline{V}$ are the complements of U, V. This is then the regressive product of U and V.

If

$$U = \delta^{i_1 \dots i_p}_{r_1 \dots r_p} \quad u_{1/i_1} \dots u_{p/i_p}$$

$$V = \delta^{j_1 \dots j_q}_{s_1 \dots s_q} \quad v_{1/j_1} \dots v_{q/j_q}$$

Then $(\overline{U}\ V) = \epsilon^{r_1 \dots r_p\, s_1 \dots s_{n-p}} \quad \delta^{i_1 \dots i_p}_{r_1 \dots r_p} \quad (u_p)\, \delta^{j_1 \dots j_q}_{s_1 \dots s_q} (v_q)$

where $(u_p) = u_{1/i_1} \dots u_{p/i_p}$, $(v_q) = v_{1/j} \dots v_{q/j_q}$.

It is easily shown that

$$\epsilon_{r_1 \dots r_p\, s_1 \dots s_{n-p}} \quad \delta^{i_1 \dots i_p}_{r_1 \dots r_p} = (n-p)! \quad \epsilon^{i_1 \dots i_p \quad s_1 \dots s_{n-p}}$$

and hence

$$(\overline{U}V) = (n-p)! \quad \epsilon^{i_1 \dots i_p \quad s_1 \dots s_{n-p}} (u_p) \quad \delta^{j_1 \dots j_q}_{s_1 \dots s_q} (v_q) \tag{7}$$

$$= (n-p)!(p+q-n)! \left[\epsilon^{i_1 \dots i_p\, j_1 \dots j_{n-p}} u_{1/i} \dots u_{p/i_p} v_{1/j_1} \dots v_{n-p/j_{n-p}} \right]$$

$$\left[\delta^{j_{n-p+1} \dots j_q}_{s_{n-p+1} \dots s_q} v_{n-p+1/j_{n-p+1}} \dots v_{q/j_q} \right]$$

The first bracket is seen to be a scalar and consists of the

complement of U and $n-p$ vectors from V and the second bracket is the outer product of the balance of V. This is known as the *rule of the excluded middle*. It is easily seen that the product can also be expressed in terms of the vectors u_i.

The quantities U, V, W so far used are all the outer progressive product of vectors and are called simple geometric quantities. The sum of two or more quantities of the same dimensions cannot necessarily be expressed as the progressive product of vectors and when they cannot are called *complex* geometric quantities.

6. **Other products.** The summations so far considered have extended over all the indices of at least one of the factors and we will now consider the significance when this is not the case. Let us first investigate

$$V^{s_1 \cdots s_{n-p}}_{r_{p+1} \cdots r_q} = \epsilon^{r_1 \cdots r_p \; s_1 \cdots s_{n-p}} \; U_{r_1 \cdots r_q} \tag{8}$$

where $q > p$ and U is a *simple* geometric quantity, we see at once that V is an alternating function of s and of r and if we express it in covariant coördinates $V_{t_1 \cdots t_p s_{p+1} \cdots s_q}$ it is not an alternating function of t and r together and hence is not a simple geometric quantity. As a transformation, V has the following properties. A covariant vector v_{r_p} by multiplication with V is transformed into a quantity of the same character as V but having one less covariant index. A geometric quantity $U_{s_1 \cdots s_p}$ if multiplied by V is transformed into a simple geometric quantity $W_{r_{p+1} \cdots r_q}$ and is the part of $U_{r_1 \cdots r_q}$ common to $U_{s_1 \cdots s_p}$. The product of V by a contravariant simple geometric quantity $U^{r_{p+1} \cdots r_q}$ gives the complement of the part of $U_{r_1 \cdots r_p}$ which is completely perpendicular to $U^{r_{p+1} \cdots r_p}$.

There is an invariant connected with V, *viz.*, the complete contraction, *i.e.*

$$V^{r_{p+1} \cdots r_q \; s_{q-p+1} \cdots s_{n-p}}_{r_{p+1} \cdots r_q} = V^{s_{q-p+1} \cdots s_{n-p}}$$

and is the complement of U.

Next consider the product

$$W_{t_1 \dots t_q}^{s_1 \dots s_k} = U_{r \dots r_p t_1 \dots t_q} V^{r_1 \dots r_p s_1 \dots s_k} \tag{9}$$

This is again an alternating function of s alone or of t alone but is not an alternating function of all indices when W is expressed in terms of covariant coördinates. This is of character similar to (8). Let us first investigate the condition for the vanishing of W. If this is to happen it is obvious that the $p+k$ indices of V must contain less than p common to the indices of U, which means that V must contain a space of $k+1$ dimensions completely perpendicular to U, and likewise U must contain a space of $q+1$ dimensions completely perpendicular[6] to V.

Now suppose that V contains a space of k dimensions completely perpendicular to U and U, a space of q dimensions completely perpendicular to V. Since we can choose the coördinates in the space V arbitrarily let us choose k mutually perpendicular vectors in the first and q in the second as coördinate lines. Then (9) can be written

$$W = (U_{r_1 \dots r_p} V^{r_1 \dots r_p}) u_1 u_2 \dots u_q \; u^{q+1} \dots u^{q+k}$$

where u^{q+1} means the contravariant expression of u_{q+1}, etc. The parenthesis is an invariant. Hence W is the *indeterminate* product of the part of U completely perpendicular to V and the part of V completely perpendicular to U. This is called the indeterminate product because there is no relation between $u_1 u_2 \dots a_q \, u^{q+1} \dots u^{q+k}$ and $u^{q+1} \dots u^{q+k} u_1 \dots u_q$. An invariant of W is the outer product

$$K^{i_1 \dots i_{p+q}} = \delta^{i_1 i_2 \dots i_{p+q}}_{s_1 \dots s_p \, t_1 \dots t_q} W_{s_1 \dots s_k \, t_1 \dots t_q}$$

where

$$W_{s_1 \dots s_k \, t_1 \dots t_q} = a_{s_1 j_1} a_{s_2 j_2} \dots a_{s_k j_k} W_{t_1 \dots t_q}^{j_1 \dots j_k}.$$

This invariant is the space determined by the part of U perpendicular to V and the part of V perpendicular to U. If p is decreased $K=0$. Next suppose U, V have a space of p dimensions in common and that we choose coördinates so that

$$U = u_1 u_2 \dots u_p \; U_{t_1 \dots t_q}$$

$$V = u^1 u^2 \dots u^p \; V^{s_1 \dots s_k}$$

[6] cf. Struik loc. cit. page 35.

where $U_{t_1 \dots t_q}$ is the part of U completely perpendicular to the common space and similarly for $V^{s_1 \dots s_k}$ then

$$W = (u_1 \dots u_p u^1 \dots u^p)\ U_{t_1 \dots t_q}$$

and is the indeterminate product of $U_{t_1 \dots t_p} V^{s_1 \dots s_k}$. In this case K is the space determined by the parts of U and V respectively perpendicular to the common intersection. It is obvious that $K=0$ if the intersection is of higher dimensions than p. Hence we have the theorems. If $K=0$, *either U contains a space of at least $q+1$ dimensions perpendicular to V or U and V intersect in a space of more than p dimensions.*

If $K=0$, when $p=m+1$, but $K \neq 0$ when $p=m$ then U contains a space of k dimensions completely perpendicular to V.

If $K=0$, when $p=m-1$ but $K \neq 0$ when $p=m$ then U and V have a space of m dimensions in common.[7]

Equation (9) is equivalent to

$$W = \epsilon_{r_1 \dots r_p t_1 \dots t_{n-p}}\ U^{t_{q+1} \dots t_n - p}\ V^{r_1 \dots r_p\ s_1 \dots s_k}$$

where $\overline{U}$ is the complement of U.

This last form suggests the more general form

$$P = \epsilon_{r_1 \dots r_m s_1 \dots s_k t_1 \dots t_{n-m-k}}\ U^{r_1 \dots r_p}\ V^{s_1 \dots s_q}$$

where $m<q$, $k<p$. P is an alternating function of t, r and s separately but not all three together. We shall investigate only the condition for the vanishing of P. Choose coördinates so that U is the outer product of u^1, $u^2 \dots u^p$. Then the condition for the vanishing of P is any choice of k indices of V must contain at least one in common with any set of m numbers chosen arbitrarily from the first p numbers. Then the indices of V must contain $p+q-m-k+1$, at least, of the first p integers. Then Then $P=0$ if U and V have a space of $p+q-m-k+1$ dimensions in common.

[7] See Moore and Phillips, Proc. Nat. Acad. Vol. 6, 1920.

7. **Dyadics.** Transformations of vectors into vectors can be accomplished in several ways, *e.g.*

$$\phi_r^s = \Sigma v_{i/r} u_i^s$$

$$\psi_{rs} = \sum_i^s v_{i/r} u_{i/s}$$

$$\chi_{r, s_1 \ldots s_{n-1}} = v_{i/r} U_{i/s_1 \ldots s_{n-1}}$$

The product under the summation sign is the indeterminate product. By contraction, ϕ transforms a covariant vector into a covariant vector and a contravariant into a contravariant, while χ transforms a vector into a space of $n-1$ dimensions. These three dyadics are related as follows

$$\psi_{rs} = \phi_r^t a_{st}, \chi_{rs_1 \ldots s_{n-1}} = \phi_r^s \epsilon_{ss_1 \ldots s_{n-1}}.$$

We will, therefore, confine our attention to ϕ_r^s.

If λ_i is a set n mutually orthogonal unit vectors, *i.e.*, if

$$\lambda_{i/r}\lambda_j^r = \begin{cases} 0, \text{ if } i \neq j \\ 1, \text{ if } i = j \end{cases}$$

$$\sum_i \lambda_{i/r}\lambda_i^s = \begin{cases} 0, \text{ if } r \neq s \\ 1, \text{ if } r = s \end{cases}$$

$$I_r^s = \sum_i \lambda_{i/r}\lambda_i^s$$

is the idemfactor, *i.e.*, the dyadic which leaves vectors unchanged. For any vector can be expressed in terms of λ_i,

$$v_r = \sum_i a_i \lambda_{i/r}$$

from which it is obvious that

$$I_s^r v_r = v_r$$

We have, however, already found a more convenient form for this idemfactor, *viz.* δ_r^s. Now all the properties of the Gibbs dyadic follow immediately by contraction with ϕ and with δ.

The 2, 2 dyadic can be written

$$\phi_{rs}^{iu} = \sum_{i,j} u_{ij/rs} v_{ij}^{iu}$$

where $u_{ij/rs}$, v_{ij}^{tu} are outer products. If u and v are expressed in terms of the λ's then the dyadic

$$\delta_{rs}^{tu} = I_{rs}^{tu} = \sum_{ij} \lambda_{ij/rs} \lambda_{ij}^{tu}$$

where $\lambda_{ij/rs} = \delta_{rs}^{kl} \lambda_{i/r} \lambda_{k/s}$ and $\lambda_{ij}^{tu} = \delta^{tu} \lambda_i^k \lambda_j^l$,

is the (2, 2) idemfactor. For

$$\Sigma_{ij} \lambda_{ij/rs} \lambda_{ij}^{tu} = \delta_{rs}^{tu}$$

$$\Sigma_{rs} \lambda_{ij/rs} \lambda_{kl}^{rs} = \delta_{ij}^{kl}$$

Similarly the (m, m) idemfactor is

$$\delta_{i_1 \ldots i_m}^{j_1 \ldots j_m} = I_{i_1 \ldots i_m}^{j_1 \ldots j_m}$$

and the contraction with any geometric quantity of the mth order leaves that quantity unchanged.

In what has gone before I have not stopped to justify the use of the word product but from the familiar properties of covariant and contravariant systems the operations which I have designated as products are seen to satisfy the distributive law and therefore according to Grassmann can be called products. It is here to be noted that all these products are defined in terms of one operation only, *viz.*, *contraction*.

If the vectors here considered lie in a euclidean space and the coördinates are cartesian then the distinction between covariant and contravariant is lost and all the operations run as in ordinary Grassmann analysis.

THE FUNDAMENTAL THEOREM OF ALMOST PERIODIC FUNCTIONS OF TWO VARIABLES[1]

By PHILIP FRANKLIN

1. Introduction

In this paper we shall extend the fundamental theorem of almost periodic functions of one variable, proved by Harold Bohr in his memoir on these functions[2] to the almost periodic functions of two variables defined by us in a previous paper.[3] We shall also prove some of the consequences of this theorem. The results of our earlier paper, to which we shall frequently refer as "Elementary Theory," will be assumed *in toto*.

The methods of this, as well as those of our previous paper, are capable of extension to the case of almost periodic functions of any finite number of variables. Since writing this paper, we have been informed that most of Bohr's results have been extended not only to functions of n variables, but also to those of an infinite number of variables by Dr. Bochner, in a paper to appear in the *Mathematische Annalen*.

We already know that, if $f(x, y)$ is an almost periodic function of two variables, with an infinite number of non-vanishing coefficients, $A_{mn} = \alpha(\lambda_m, \mu_n)$, then $\sum_1^\infty \sum_1^\infty |A_{mn}|^2$ converges and:

$$\sum_1^\infty \sum_1^\infty |A_{mn}|^2 \leq M\{\,|f(x, y)|^2\,\}.^4$$

We wish to prove that the equality sign holds in all cases. Following Bohr, we shall throw this back to the corresponding known theorem for periodic functions. We do this by approximating our almost periodic function by a sequence of doubly periodic functions.

[1] Presented to the American Mathematical Society, October 31, 1925.

[2] Zur Theorie der fast periodischen funktionen, Acta Mathematica, 45: pp. 29–127. (1924).

[3] The Elementary Theory of Almost Periodic Functions of Two Variables, this Journal, Vol. V, No. 1, (1926), pp. 40–54.

[4] Elementary Theory, p. 54, Theorem XIII.

The approximation is to be such that, as we let the periods of these functions increase indefinitely, the average of the square of the absolute value of these functions over a period parallelogram approaches $M\{|f(x, y)|^2\}$, while simultaneously, the sum of the squares of the absolute values of the coefficients of the approximating functions approaches the corresponding sum for the almost periodic function. To prove this latter property, we shall need certain properties of the displacement vectors of our almost periodic functions, and shall accordingly start with them. Our theorems are, of course, motivated by the later applications, but are given in their logical order, rather than in one which makes their *raison d'etre* obvious.

2. Properties of displacement vectors.

We begin with an analysis of the displacement vectors, and recall that the vector (s, t) belongs to ϵ for the function $f(x, y)$, provided that:

$$|f(x+s, y+t)-f(x, y)| \leq \epsilon.$$

We will denote by $e(s, t)$ the *minimum* value of ϵ for which s, t is a displacement vector. It follows immediately from the definition that:

$$e(s_1+s_2, t_1+t_2) \leq e(s_1, t_1)+e(s_2, t_2).$$

Also, since with another notation this becomes:

$$e(s_1, t_1) \leq e(-s_2, -t_2)+e(s_1+s_2, t_1+t_2),$$

and obviously

$$e(-s_2, -t_2)=e(s_2, t_2),$$

we have:

$$e(s_1+s_2, t_1+t_2) \geq e(s_1, t_1)-e(s_2, t_2).$$

We may now formulate:

Theorem I. *To every $\epsilon>0$, a length $L(\epsilon)$ may be found such that in each square of side L with sides parallel to the axes, there is at least one integral displacement vector, p, q with $e(p, q) \leq \epsilon$.*

To prove this theorem, we form an auxiliary doubly periodic function $P(x, y)$ defined as follows: Its value at the origin is 2ϵ, and at points inside a circle about the origin of radius $\delta<\frac{1}{4}$,

its value is $(\delta-r)2\epsilon/\delta$, where r is the distance from the origin. It is periodic in x and y, with periods unity in each case. This defines it in all the circles of radius δ with centers at integral points. Outside these circles its value is zero. We note that it is continuous and hence, being doubly periodic, is almost periodic. We take the δ so small that if s, t is a displacement vector for $f(x, y)$ for $\epsilon/2$, and $|s'-s|<\delta$, $|t'-t|<\delta$, then s', t' is a displacement vector for ϵ.[5] Now take a common displacement side a for $f(x, y)$ and $P(x, y)$ belonging to $\epsilon/2$[6], *i.e.*, such that in each square of this side there is a common displacement vector of the two, s'', t''.

$L(\epsilon)$ may be taken as $a+1$. For, in any square of side $a+1$ we may take the concentric square of side a, and in this find a displacement vector for $\epsilon/2$ for both $P(x, y)$ and $f(x, y)$. But, since it belongs to $P(x, y)$, from the construction of that function it must be within a distance δ of an integral point. This integral point is inside the square of side $a+1$, and, from the choice of δ, is a displacement vector for ϵ for $f(x, y)$.

Theorem II. *For every pair of real numbers $\nu\neq 0$, $\nu'\neq 0$, $\epsilon>0$, and $\eta>0$, there is an $L'(\nu, \nu', \epsilon, \eta)$ such that in every square of side L', with sides parallel to the axes, there is at least one displacement vector with integral components, p, q, such that $e(p, q)\leq\epsilon$, and if it be placed with its initial point at the origin, its terminus will be at distance less than η from some point whose coördinates are integral multiples of ν, ν'.*

We form an auxiliary periodic function $P(x, y)$ as in the proof of Theorem I, taking $\delta<\eta/2$, $\nu/2$, $\nu'/2$, and also a second auxiliary function $Q(x, y)$ defined in the same way inside the circle of radius δ with center at the origin, and with periods ν in x and ν' in y.

If a is a common displacement side for the three functions $f(x, y)$, $P(x, y)$ and $Q(x, y)$ belonging to $\epsilon/2$, we may take $a+1$ as the number L'. For, in any square of side $a+1$ with sides parallel to the axes, we may find in the concentric square of side a, a displacement vector for $f(x, y)$, $P(x, y)$ and $Q(x, y)$ belonging

[5] Elementary Theory, p. 42, Corollary, Theorem II.
[6] Elementary Theory, p. 43, Theorem III, Correction p.

to $\epsilon/2$. From the construction of $P(x, y)$ and $Q(x, y)$ it is within a distance δ of a point with integral coördinates, and a point whose coördinates are integral multiples of ν, ν'. The integral point is the point which gives the displacement vector whose existence we are trying to prove. It is clearly inside the square of side L', is within a distance $2\delta<\eta$ of a point with coördinates integral multiples of ν, ν' and, from the choice of δ, is a displacement vector for $f(x, y)$ belonging to ϵ.

We have now shown the existence of integral displacement vectors for any ϵ. We designate the collection of such vectors by E_ϵ. We shall say that *a function is "almost periodic up to* $1/Q$," *where* Q *is an integer if, for any* $\rho<\epsilon$, *there exists a number* I_0 *such that if* s', t' *is a vector of the collection* E_ρ, *and hence in* E_ϵ, *and all the* s, t *belonging to* E_ϵ *lying inside a square with side* $I>I_0$ *and sides parallel to the axes are displaced by* s', t', *more than* $(1-1/Q)$ *of the new vectors belong to* E_ϵ.

In terms of this definition, we may state:

Theorem III. *If* ϵ_0 *and* Q *are given, then there exists an* $\epsilon\leq\epsilon_0$, *such that* E_ϵ *is almost periodic up to* $1/Q$.

From the statement of the theorem and the definition just given, we see that we must determine quantities ϵ, $\rho<\epsilon$, and I_0. We select ϵ a number between $\epsilon_0/2$ and ϵ_0, and take $\rho=\epsilon_0/2N$, where N is an integer greater than 8 to be specified more precisely later. We divide the interval $\epsilon_0/2<z<\epsilon_0$ into N equal parts ρ. We shall say that s, t is in the rth restricted interval if $e(s, t)$ is in the interval $\epsilon_0/2+(r-1)\rho<z\leq\epsilon_0/2+r\rho$. Now consider a displacement vector s, t which is in the r_0th restricted interval, and consider the result of displacing this vector by a vector s', t' in the collection E_ρ. From our earlier inequalities, we obviously have:

$$\epsilon_0/2+(r_0-2)\rho<e(s', t')-e(s, t)\leq e(s+s', t+t')$$
$$\leq e(s', t')+e(s, t)\leq\epsilon_0/2+(r_0+1)\rho.$$

That is, the changed vector is in either the r_0-1, the r_0 or the r_0+1th restricted interval, we say it is in the extended r_0th interval.

Now take $I_0>2L(\epsilon_0/2)$, and subject to certain other conditions

to be stated later. For every integral vector in the square with side I_0 having one corner at the origin, which is in E_{ϵ_0}, but not in $E_{\epsilon_0/2}$, we shall have $\epsilon_0/2 < e(s, t) \leq \epsilon_0$. We find the restricted interval in which each of these numbers $e(s, t)$ lies, and select that extended interval which contains the least number of them. Let it be the Rth. Since the number of extended intervals which do not overlap is the greatest integer in $N/3$, at least $(N-2)/3$, and this is greater than $N/4$, since $N > 8$, the Rth extended interval must contain less than $I_0^2/N/4 = 4I_0^2/N$. We take $\epsilon = \epsilon_0/2 + R\rho$, the largest number in the restricted Rth interval.

We shall now show that, taking I_0, N and ϵ subject to the conditions already imposed, we may take I_0 and N so large, and hence $\rho = \epsilon_0/2N$ so small, that they satisfy the requirements of the definition of " almost periodic up to $1/Q$." Let I be an arbitrary number greater than I_0, and s, t an arbitrary vector of the collection E_ρ in the square $x_1 \leq x \leq I + x_1$, $y_1 \leq y \leq I + y_1$. Let A be the number of vectors s', t' in E_ϵ and in this square which, on being displaced by the vector s, t do not go into vectors of the collection E_ϵ. These will all have to be ones for which $e(s', t')$ is in the restricted Rth interval, from our choice of ϵ. To apply our knowledge about the vectors in squares of side I_0 to those of side I, we proceed to fill up the square of side I with smaller squares in the following way. We take a square of side $L(\rho)$ at the lower left hand corner of the square of side I, find an integral displacement vector belonging to ρ in it, and form a square of side I_0 with this point as lower left hand corner, and sides parallel to the axes. Touching this square, and having the lower side of the I square as one side, we take a square of side $L(\rho)$, find an integral displacement vector belonging to ρ in this square, and construct a second square of side I_0 with sides parallel to the axes and this point as lower left hand corner. Touching this square, and with the lower side of the I square as one side, we take a third square of side $L(\rho)$, and repeat the process. We thus obtain a series of squares of side I_0, the last of which may lie only partly in the I square, all in a strip of width $I_0 + L(\rho)$ along the bottom of the I square. We now take a square of side $L(\rho)$ with the left hand side of the I square as one side, and the top of the strip just mentioned as another, find an integral displacement vector belong-

ing to ρ in it, and form a square of side I_0 with it as left hand corner. We form further squares of side $L(\rho)$, and I_0 to form a second strip of I_0 squares. We keep up this process until our strips fill up the I square. The I_0 squares of the last strip in general will lie only partly in the I square. We note that the number of I_0 squares, wholly or partly inside the I square, is at most $\left(\frac{I}{I_0}+1\right)^2$, which is less than $4I^2/I_0^2$. The part of the I square not in these I_0 squares may be looked on as included in a set of areas, one for each I_0 square, formed by removing from a square of side $I_0+L(\rho)$, one of side I_0.

Now consider the integral vectors belonging to ϵ in the square I, which, on being displaced by some integral vector belonging to ρ do not go into integral vectors belonging to ϵ. They must be ones for which $e(s, t)$ is in the restricted Rth interval. Since there is a displacement vector belonging to ρ which takes each of the I_0 squares into that with the origin at one corner, all such vectors in an I_0 square go under this displacement into vectors in this latter square for which $e(s, t)$ is in the extended Rth interval, and hence their number is less than $4I_0^2/N$. As there are less than $4I^2/I_0^2$ I_0 squares, the total number in I_0 squares is at most $16\ I^2/N$. For vectors of this type not in I_0 squares, we note that the number of integral points in a square of side $I_0+L(\rho)$, including the bottom and left hand side, but not the top or right hand side, is at most $[I_0+L(\rho)]^2$, while the number in a square of side I_0 including the boundary is at least $(I_0-1)^2$. Thus the number of integral points in an area formed by removing a square of side I_0 from one of side $I_0+L(\rho)$ is at most $[I_0+L(\rho)]^2-[I_0-1]^2=[L(\rho)+1]\ [2I_0+L(\rho)-1]$. As the part of the I square not in I_0 squares may be included in a number less than $4I^2/I_0^2$ of such areas, the number of integral points in this area, and hence *a fortiori* the number of integral vectors of a special character, will be less than $4I^2/I_0^2[L(\rho)+1]\ [2I_0+L(\rho)-1]$. Thus if A is the total number of vectors in the I square in E_ϵ which, under displacement by a vector in E_ρ do not go into vectors in E_ϵ, we have:

$$A<4I^2\left\{4/N+2[L(\rho)+1]/I_0+[L(\rho)^2-1]/I_0^2\right\}.$$

Next consider the total collection of integral vectors belonging to

ϵ, *i.e.* in E_ϵ, in the I square. Since there is an integral vector belonging to $\epsilon_0/2$, and hence to ϵ since $\epsilon > \epsilon_0/2$, in each square of side $L(\epsilon_0/2)$; the number of such vectors is greater than $\left[\frac{I}{L(\epsilon_0/2)} - 1\right]^2$. But, as $I > I_0 > 2L(\epsilon_0/2)$, this is greater than $I^2/4L(\epsilon_0/2)^2$. Thus, if B is the total number of vectors in the I square in E, we have:

$$B > I^2/4L(\epsilon_0/2)^2.$$

By combining this with the relation for A above, we have:

$$\frac{A}{B} < \frac{64\ L(\epsilon_0/2)^2}{N} + \frac{32\ L(\epsilon_0/2)^2[L(\rho)-1]}{I_0} + \frac{16\ L(\epsilon_0/2)[L(\rho)^2-1]}{I_0^2}.$$

By choosing N large, we may make the first term small, and then when $\rho = \epsilon_0/2N$ is fixed and hence $L(\rho)$, we may choose I_0 to make the other terms small. Hence we may make the sum less than $1/Q$, and thus find the quantities whose existence was stated in the theorem.

For later purposes, we shall need to know how the frequencies behave when a set are multiplied by a constant. Since we use them as exponents, we are only concerned with their values when reduced modulo 2π. We introduce the following notation. If a vector x, y is such that $x+y$ is numerically less than or equal to $\pi/6$, modulo 2π, we say it is in the *zero class*. If $x+y$ is numerically greater than $\pi/3$, modulo 2π, we say it is in the *restricted π class*. If $x+y$ is numerically greater than $\pi/6$, modulo 2π, we say it is in the *extended π class*. The sum of two vectors is clearly in the extended π class, if one of them is in the zero class, and the other in the restricted π class. We have:

Theorem IV. *In order that the real numbers λ, μ, for any $N > N_0(\lambda, \mu)$ consecutive integral displacement vectors belonging to ϵ, s_1, t_1, s_2, t_2 . . . s_N, t_N, where by consecutive we mean the set consists of all the members of E_ϵ in a square, have the property that more than $\frac{1}{4}$ of the vectors λs_1, μt_1, λs_2, μt_2, . . . λs_N, μt_N are in the extended π class, i.e., are numerically greater than $\pi/6$, modulo 2π, it is sufficient that λ, μ have the:*

"Property π": In the collection E_ρ $(\rho < \epsilon)$ a vector s, t exists such that λs, μt is in the restricted π class.

Our proof depends on the two facts that, for any integer Q, we may find an I_0 by Theorem III such that if the consecutive set includes the vectors in a square of side I_0, when they are displaced by s, t except for $1/Q$ of them they will go into vectors of the set E_ϵ, and if a vector λs_i, μt_i is in the zero class, $\lambda(s_i+s)$, $\mu(t_i+t)$ is in the extended π class.

Suppose λ, μ has the property π, and s, t is the vector of the set E_ρ such that λs, μt is in the restricted π class. We take $N_0=S_0^2$, where $S_0>I_0$, the I_0 of Theorem III for ρ, ϵ and $Q=4$, and $S_0>2L(\epsilon)+8s'\ L(\epsilon)^2$, s' being the larger of the pair s, t. We note that it is a consequence of this condition that:

$$\frac{S_0^2-2S_0L(\epsilon)}{4\ L(\epsilon)^2}>2S_0s', \text{ or } \tfrac{1}{4}\left[\frac{S_0}{L(\epsilon)}-1\right]^2>2S_0s'.$$

Now take N consecutive vectors, where $N>N_0$. Let the number of vectors λs_i, μt_i in the extended π class be A, and the number in the zero class be B. We have $B=N-A$. We must prove that $A>N/4$. Since the vectors are consecutive, they consist of all the members of E_ϵ in a square. Let the side of one such square be $S+1$, S an integer. Draw a square of side $S-s'+1$ in the $S+1$ square, having two sides in common with it, the common vertex being such that if it is taken as one extremity of the vector s, t, the other extremity will lie inside the square. Let B_1 be the number of vectors in the class B in this latter square. The total number of integral points in this square is $(S-s')^2$, while the number in the large square is S^2. The difference between these numbers is clearly greater than or equal to $B-B_1$, so we have:

$$B-B_1\leq 2Ss'-s'^2<2Ss'.$$

But as $S>S_0>2L(\epsilon)+8\ s'\ L(\epsilon)^2$, we have

$$2Ss'<\tfrac{1}{4}\left[\frac{S}{L(\epsilon)}-1\right]^2<\frac{N}{4},$$

since there is at least one member of E_ϵ in each square of side $L(\epsilon)$. Thus we have:

$$B-B_1<N/4, \quad \text{or} \quad B_1>B-N/4.$$

But, if any vector s_i, t_i, of the class B_1 is displaced by s, t, it will give a vector in the $S+1$ square, and since λs_i, μt_i, is in the zero class, while λs, μt is in the restricted π class, $\lambda(s_i+s)$, $\mu(t_i+t)$ will be in the extended π class, and hence if the displaced vector s_i+s, t_i+t is in E_ϵ, it will be in the class A. But, since $N>N_0>I_0^2$, the number of such displaced vectors not in E_ϵ will be at most N/Q or $N/4$, and we have:

$$A>B_1-N/4,$$

or

$$A>B-N/2=N/2-A, \quad A>N/4,$$

which we were to prove.

Our next two theorems discuss the distribution of vectors with the property π.

Theorem V. *If λ_0, μ_0 has the property π, there is a square neighborhood of λ_0, μ_0, $|\lambda-\lambda_0|<\kappa$, $|\mu-\mu_0|<\kappa$, such that:*

(1) *Every vector in this neighborhood has the property π, and for every sufficiently great N, more than $\frac{1}{4}$ of the N vectors λs_i, μt_i formed from the N consecutive integral vectors s_i, t_i belonging to ϵ are in the extended π class.*

(2) *For a sufficiently great N, this holds uniformly with respect to all the points in the interval, i.e., there is an $N_0(\kappa)$, independent of λ, μ such that $N>N_0(\kappa)$ insures the property for all λ, μ with $|\lambda-\lambda_0|<\kappa$, $|\mu-\mu_0|<\kappa$.*

If λ_0, μ_0 has the property π, by the definition, there is a vector s_0, t_0 such that $\lambda_0 s_0$, $\mu_0 t_0$ is in the restricted π class, *i.e.*, such that $\lambda_0 s_0+\mu_0 t_0$ is, modulo 2π, numerically $>\pi/3$. We may take κ so small that λs_0, μt_0 is in the restricted π class, for all λ, μ in a κ neighborhood of λ_0, μ_0. Also, by Theorem IV, we may take $N_0(\kappa)>I_0^2$, $[2L(\epsilon)+8s'L(\epsilon)^2]$,[2] which restrictions are independent of λ, μ.

Theorem VI. *The set of points λ_1, μ_1 which do not have the property π, (not a nul set since $2\pi m$, $2\pi n$, with m, n integers are in the set) have no limit point, and hence have only a finite number in any finite region.*

We will show that no point λ_1, μ_1 can be a limit point of the set

by exhibiting a square $|\lambda-\lambda_1|<h$, $|\mu-\mu_1|<h$, such that all the points in this interval except perhaps λ_1, μ_1 itself have the property π. In Theorem V we have already treated the case where λ_1, μ_1 has the property π. Hence we may assume in the proof which follows that λ_1, μ_1 has not the property π. That is, there is no vector s, t in E_ρ such that $\lambda_1 s$, $\mu_1 t$ is in the restricted π class. But there are vectors s, t in E_ρ such that $\lambda_1 s$, $\mu_1 t$ is in the zero class, in particular, we will show that there is one of them in every square of side $L'=L'(\lambda_1, \mu_1)$.

1. If $\lambda_1, \mu_1=0$, 0 every $\lambda_1 s$, $\mu_1 t=0$, 0 and is in the zero class. Thus we may take L' as the $L'(\rho)$ of Theorem I.

2. If $\lambda_1\neq 0$ and $\mu_1\neq 0$, we must show that a length L' exists such that in every square of side L' there is a vector s, t of E_ρ, for which $\lambda_1 s+\mu_1 t$ differs from an integral multiple of 2π by at most $\pi/6$. This condition will certainly be satisfied if the distance of the vector s, t from a point whose coördinates are integral multiples of $2\pi/\lambda_1$, $2\pi/\mu_1$ is less than $\dfrac{\pi}{6(|\lambda_1|+|\mu_1|)}$. Thus we use this as the η of Theorem II, and take L' as the $L'\left(2\pi/\lambda_1,\ 2\pi/\mu_1,\ \rho,\ \dfrac{\pi}{6(|\lambda_1|+|\mu_1|)}\right)$ of that theorem.

3. If $\lambda_1=0$, $\mu_1\neq 0$, we may clearly take L' as the $L'\left(\nu, 2\pi/\mu_1, \rho, \dfrac{\pi}{6|\mu_1|}\right)$ of Theorem II, where ν is any number not zero, while if $\lambda_1\neq 0$, $\mu_1=0$, we may use $L'\left(2\pi/\lambda_1,\ \nu',\ \rho\ \dfrac{\pi}{6|\lambda_1|}\right)$ with ν' any number not zero.

We now put $h_1=\pi/4L'$, using the $L'(\lambda_1, \mu_1)$ whose existence we have just proved, and take $0<h<h_1$. Every point in the square $|\lambda-\lambda_1|<h$, $|\mu-\mu_1|<h$ has the property π. For, in every square of side L' there is a vector s, t in E_ρ for which $\lambda_1 s$, $\mu_1 t$ is in the zero class. In particular, consider a series of squares L' on a side, with sides parallel to the axes, and one vertex at the point $x=y=kL'$, k an integer taking on in succession the values 1, 2, 3, ... Let s^k, t^k be the vector in the kth square. We have

$$2\leq(s^{k+1}+t^{k+1})-(s^k+t^k)\leq 4L',$$

and hence:

$$0<2h\leq h(s^{k+1}+t^{k+1})-h(s^k+t^k)\leq 4L'h<\pi.$$

This shows that the numbers $h(s^k+t^k)$ increase by positive amounts less than π. Hence there must be one which, modulo 2π, is in the open interval $(\pi/2, 3\pi/2$. But as $\lambda_1 s^k$, $\mu_1 t^k$ is in the zero class, we have $\lambda_1 s^k+\mu_1 t^k$, modulo 2π, numerically less than or equal to $\pi/6$. Hence $(\lambda_1+h)s^k+(\mu_1+h)t^k$ is modulo 2π, in the open interval $\pi/3$, $5\pi/3$, which shows that the vector λ_1+h, μ_1+h, that is any vector in the square $|\lambda-\lambda_1|<h_1$, $|\mu-\mu_1|<h_1$, has the property π.

We are now able to prove the property of displacement vectors which we shall need in proving our fundamental theorem.

Theorem VII. *Given $\epsilon>0$, $\Omega>0$, it is possible to find, in the square $|\lambda|<\Omega$, $|\mu|<\Omega$, M points $M(\epsilon, \Omega)$: P_1, P_2, $\ldots$ P_M such that if $\omega>0$ be given, a set of displacement vectors belonging to ϵ; s_1, t_1, s_2, t_2, $\ldots$ s_N, t_N, N depending on ω, can be found with the following property. For every λ, μ inside the square $|\lambda|<\Omega$, $|\mu|<\Omega$, and outside the M squares $|\lambda-\lambda_i|<\omega$, $|\mu-\mu_i|<\omega$, where $(\lambda_i, \mu_i)=P_i$, more than $1/4$ of the N products $\lambda s_j+\mu t_j$ are modulo 2π, numerically greater than $\pi/6$.*

By Theorem VI there are only a finite number of points in the square $|\lambda|<\Omega$, $|\mu|<\Omega$, not having the property π. We take these as the points $P_i=(\lambda_i, \mu_i)$ and construct the squares $|\lambda-\lambda_i|<\omega$, $|\mu-\mu_i|<\omega$. Every point outside these squares, and inside the Ω square has the property π. Hence by Theorem V, it may be surrounded by a κ neighborhood such that every point (λ, μ) in this neighborhood has the property π, and for a sufficiently great N, $>N_\rho$, more than $1/4$ of the N quantities $\lambda s_j+\mu t_j$ are, modulo 2π, numerically greater than $\pi/6$, if s_j, t_j are any N consecutive vectors. By the Heine Borel theorem we may find a finite subset of these κ neighborhoods which completely cover the region consisting of points inside the Ω square, and outside the M ω squares. This gives a finite number of values of N_ρ, and we take the largest such value as the N of the theorem, and N consecutive integral displacement vectors belonging to ϵ, as the vectors s_1, t_1, s_2, t_2, $\ldots$ s_N, t_N.

Corollary. The vectors s_j, t_j may be taken so that $s_j>0$, $t_j>0$.

3. Approximating doubly periodic functions.

In proving the fundamental theorem for almost periodic functions of two variables, we shall use the corresponding known theorem for doubly periodic functions. To apply this theorem, we shall define, for a given almost periodic function, $f(x, y)$, approximating doubly periodic functions $f_T(x, y)$, by the following relations:

$$f_T(x, y)=f(x, y) \qquad 0\leq x<T, \qquad 0\leq y<T,$$
$$f_T(x+T, y)=f_T(x, y+T)=f_T(x, y).$$

That is, $f_T(x, y)$ is a doubly periodic function, with both periods equal to T, which, in the first period parallogram, here a square, agrees with $f(x, y)$. In our use of these functions, we shall generally be concerned with a sequence of them for increasing values of T. In general these approximating functions are not almost periodic in our sense, since they are discontinuous.

For doubly periodic functions, summable and with summable square, Parseval's theorem is known to hold.[7]. That is:

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}|a_{mn}|^2=1/T^2\int_0^T\int_0^T|f_T(x, y)|^2dx\,dy,$$

where

$$a_{mn}=1/T^2\int_0^T\int_0^T f_T(x, y)e^{-imKx-inKy}dx\,dy, \quad K=2\pi/T.^{8}$$

It will be noticed that, as T becomes infinite, we have:

$$\lim_{T=\infty}1/T^2\int_0^T\int_0^T|f_T(x, y)|^2dx\,dy=\lim_{T=\infty}1/T^2\int_0^T\int_0^T|f(x, y)|^2\,dx\,dy$$
$$=M\{|f(x, y)|^2\}.^{9}$$

We shall prove relations involving $|A_{mn}|^2$ from the corresponding relations involving $|a_{mn}|^2$. For this purpose, we need to know certain properties of these latter quantities. The following three theorems enable us to derive an important property of these coefficients from Theorem VII.

[7]cf. e. g. R. Courant und D. Hilbert, Methoden der Mathematischen Physik, Berlin, 1924, p. 60.
[8]cf. Theorem XI, Elementary Theory, p. 52.
[9]cf. Theorem VI, Elementary Theory, p. 45.

Theorem VIII. *For every $\delta>0$, there exists an $\epsilon(\delta)>0$ such that, if $s>0$, $t>0$ is any displacement vector of $f(x, y)$ belonging to ϵ, and $T>T_0(\delta, \epsilon, s, t)$, the coefficents of the approximating periodic function $f_T(x, y)$, a_{mn}, satisfy the following relation:*

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}|a_{mn}|^2|1-e^{imKs+inKt}|^2<\delta, \qquad K=2\pi/T.$$

To prove this we note that the coefficients of the periodic function $f_T(x, y)$ are:

$$a_{mn}=1/T^2\int^T\int^T f_T(x, y)e^{-imKx-inKy}dx\,dy.$$

Those of the function $f_T(x+s, y+t)$ are:

$$a_{mn}'=1/T^2\int_0^T\int_0^T f_T(x+s, y+t)e^{-imKx-inKy}dx\,dy.$$

On account of the double periods, each equal to $T=2\pi/K$ of $f_T(x, y)$ and the exponential factor, this is:

$$1/T^2\int_{0-s}^{T-s}\int_{0-t}^{T-t} f_T(x+s, y+t)e^{-imKx-inKy}dx\,dy,$$

$$=1/T^2\int_0^T\int_0^T f_T(x', y')e^{-imKx'-inKy'+imKs+inKt}dx'\,dy',$$

$$=a_{mn}e^{imKs+inKt}.$$

Since the operation of taking the coefficients is linear, the coefficients for the difference of two functions is the difference of the coefficients, so that those for the function:

$$f_T(x, y)-f_T(x+s, y+t)$$

are:

$$a_{mn}-a_{mn}e^{imKs+inKt}=a_{mn}(1-e^{imKs+inKt}).$$

On applying Parseval's theorem to this function, we find:

$$1/T^2\int_0^T\int_0^T|f_T(x, y)-f_T(x+s, y+t)|^2\,dx\,dy$$

$$=\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}|a_{mn}|^2|1-e^{imKs+inKt}|^2.$$

But, since $f_T(x, y)=f(x, y)$ for $0\leq x<T$, $0\leq y<T$, and s, t is a displacement vector for $f(x, y)$ belonging to ϵ, for $0\leq x<T-s$, $\leq y<T-t$, we have:

$$f_T(x+s, y+t)=f(x+s, y+t)=f(x, y)+\theta\epsilon=f_T(x, y)+\theta\epsilon, \quad |\theta|\leq 1,$$

This shows that

$$|f_T(x, y)-f_T(x+s, y+t)|<\epsilon, \quad 0<x<T-s, \quad 0<y<T-t.$$

But, for all values of x and y, if G is an upper bound for $|f(x, y)|$ shown to exist in Theorem I, Elementary Theory, p. 42, we have:

$$|f_T(x, y)-f_T(x+s, y+t)|<2G.$$

But, as the left member of the Parseval formula may be written:

$$1/T^2\int_0^{T-t}\int_0^{T-s}|f_T(x, y)-f_T(x+s, y+t)|^2dx\,dy$$

$$+1/T^2\int_R\int|f_T(x, y)-f_T(x+s, y+t)|^2dx\,dy,$$

where R is the remainder of the T square, it is, in absolute value, less than:

$$\frac{(T-t)(T-s)\epsilon^2}{T^2}+\frac{(Tt+Ts-st)(2G)^2}{T^2}.$$

or less than:

$$\epsilon^2+\frac{(s+t)4G^2}{T}.$$

But this will be less than δ provided we take $\epsilon<\sqrt{\delta/2}$, and $T>T_0>8G^2(s+t)/\delta$. Hence the right member of the Parseval formula, which is the quantity referred to in the theorem, will be in absolute value less than δ.

We may extend this result to a finite number of vectors in:

Theorem IX. *For every $\delta>0$, there exists an $\epsilon(\delta)>0$, such that if $s_i>0$, $t_i>0$ are any q displacement vectors of $f(x, y)$ belonging to ϵ, q being an arbitrary number, and $T>T_0(\delta, \epsilon, s_1, t_1, \ldots s_q, t_q)$, the coefficients of $f_T(x, y)$ satisfy the following relation:*

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}|a_{mn}|^2\sum_{i=1}^{q}\frac{|1-e^{imKs_i+inKt_i}|^2}{q}<\delta.$$

By Theorem VIII, we may find an $\epsilon(\delta)$ and a T_{0i} such that, for $T>T_{0i}$ and $s_i>0$, $t_i>0$, any displacement vector belonging to $\epsilon(\delta)$, we have:

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}|a_{mn}|^2|1-e^{imKs_i+inKt_i}|^2<\delta.$$

If we take T_0 greater than any of these T_{0i}, these relations all hold simultaneously. Consequently, on taking the average of all these inequalities, we will have the inequality which was to be proved.

In preparation for the application of Theorem VII, we prove a theorem involving relations modulo 2π.

Theorem X. *For every $\delta'>0$, there exists an $\epsilon(\delta')$ such that, if s_i, t_i are any arbitrary displacement vectors belonging to ϵ, their number also being arbitrary, for all sufficiently great values of T, $T>T_0(\delta', \epsilon, s_i, t_i)$, we shall have:*

$$\sum_{-\infty}^{\infty}{}'\sum_{-\infty}^{\infty}{}'|a_{mn}|^2<\delta',$$

where the prime on the summation indicates that it is only taken over such values of m and n that more than $1/4$ of the numbers mKs_i+nKt_i, $K=2\pi/T$, are numerically greater than $\pi/6$, modulo 2π.

We note that, if $|z|>\pi/6$, modulo 2π, we have:

$$|1-e^{iz}|^2=|1-\cos z-i\sin z|^2=2-2\cos z>2-\sqrt{3}>1/4.$$

Consider the factor

$$\sum_{1}^{q}\frac{|1-e^{imK_{zi}+inKt_i}|^2}{q}.$$

If more than $1/4$ of the terms in the numerator have their exponents numerically greater than $\pi/6$, modulo 2π, the fraction will be greater than

$$q/4\cdot 1/4\cdot 1/q=1/16.$$

By Theorem IX, we may find an ϵ and a T_0 such that:

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}|a_{mn}|^2\sum_{i=1}^{q}\frac{|1-e^{imKs_i+inKt_i}|^2}{q}<\frac{\delta'}{16}.$$

But, from the relation found above, we have:

$$\sum_{-\infty}^{\infty}{}' \sum_{-\infty}^{\infty}{}' \frac{|a_{mn}|^2}{16} < \sum_{-\infty}^{\infty}{}' \sum_{-\infty}^{\infty}{}' |a_{mn}|^2 \sum_{i=1}^{q} \frac{|1-e^{imKs_i+inKt_i}|^2}{q}.$$

As the primed sum is less than or equal to the unprimed sum, this shows that:

$$\sum_{-\infty}^{\infty}{}' \sum_{-\infty}^{\infty}{}' |a_{mn}|^2 < \delta'.$$

We are now in a position to translate Theorem VII into a condition on the coefficients a_{mn}.

Theorem XI. *For every $\delta''>0$, and $\Omega>0$, there are a finite number, M, of points $P_i=(\lambda_i, \mu_i)$ in the square $|\lambda|<\Omega$, $|\mu|<\Omega$, such that, given $\omega>0$, for all sufficiently great T, $T>T_0(\omega)$, we have:*

$$\Sigma''\ \Sigma''\ |a_{mn}|^2<\delta'';$$

where the double primed summation extends over all values of m and n for which $\lambda=mK$, $\mu=nK$, $K=2\pi/T$, are inside the square $|\lambda|<\Omega$, $|\mu|<\Omega$, and outside the M squares $|\lambda-\lambda_i|<\omega$, $|\mu-\mu_i|<\omega$.

To prove this, let $\delta''>0$ and $\Omega>0$ be given. We must show that

$$\Sigma''\ \Sigma''\ |a_{mn}|^2<\delta'',$$

where the summation depends on P_i, ω, and Ω. We first find the $\epsilon(\delta')$ of Theorem X, taking δ' equal to the δ'' here given. Then we take this ϵ, the given ω and Ω, and use them as the given quantities of Theorem VII. We locate the points P_i of that theorem, and the set of displacement vectors s_j, t_j, taking them so that $s_j>0$, $t_j>0$, as we may by the corollary. Now, for every value of m, n included in the double primed summation, more than $1/4$ of the numbers mKs_j+nKt_j are, modulo 2π, numerically greater than $\pi/6$, by Theorem VII. Hence, these values are all included in the primed summation of Theorem X, and using that theorem we find:

$$\Sigma''\Sigma''|a_{mn}|^2\leq\Sigma'\Sigma'|a_{mn}|^2<\delta'=\delta''.$$

4. The Dependence of the coefficients of the approximating function on those of the given function.

We are aiming at the proof that

$$\sum_{1}^{\infty}\sum_{1}^{\infty}|A_{mn}|^2=M\{|f(x,y)|^2\}.$$

From Theorem XIII, Elementary Theory, p. 54, we have:

$$\sum_{1}^{\infty}\sum_{1}^{\infty}|A_{mn}|^2\leq M\{|f(x,y)|^2\}.$$

We also know, from the discussion at the beginning of section 3, that:

$$\lim_{T=\infty}\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}|a_{mn}|^2=M\{|f(x,y)|^2\}.$$

We will in consequence have proved our point if we show that

$$\sum_{1}^{\infty}\sum_{1}^{\infty}|A_{mn}|^2\geq\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}|a_{mn}|^2-\delta_T,$$

where δ_T approaches 0, as T becomes infinite.

We will prove the above relation of the sum $\Sigma\Sigma|A_{mn}|^2$ to the sum $\Sigma\Sigma|a_{mn}|^2$ by showing that, for T sufficiently large, the individual terms of the second sum behave as follows. Those with large subscripts have little effect. Those with subscripts corresponding to values mK, nK, $K=2\pi/T$, near values of λ, μ not in the set of exponents for $f(x, y)$ giving non-vanishing coefficients have little effect, while those for which the values mK, nK are near values of λ, μ in the set of exponents for $f(x, y)$ giving non-vanishing coefficients collectively give sensibly the same effect to the second sum that the corresponding non-vanishing coefficient gives to the first sum. These facts form the content of the next three theorems.

Theorem XII. *For every $\delta>0$, there exists an $\Omega>0$, and a $T_0>0$, such that, for $T>T_0$, we have:*

$$\sum_{|m|>\Omega/K,\text{ or }|n|>\Omega/K}|a_{mn}|^2<\delta, \qquad (K=2\pi/T).$$

We shall prove this theorem by using the properties of the derivative of an approximating periodic function, and hence our first task is to set up a doubly periodic function, approximating $f(x, y)$, or $f_T(x, y)$, which is in general differentiable. To do this we note that, by Theorem II, Elementary Theory, p. 42, $f(x, y)$ is uniformly continuous. We select a distance γ corresponding to $\eta < G$, where G is an upper bound for $|f(x, y)|$, in such a way that, at any two points whose distance is less than 2γ, the absolute value of the difference of the values of $f(x, y)$ at these points is less than η. Now divide the square $0 < x < T$, $0, < y < T$, by a square network with meshes γ on a side, the lines of the network being parallel to the axes, and the origin being one vertex. We erase the lines $x = N\gamma$, $y = N\gamma$, N being the greatest integer in T/γ. At the right-hand side, and top of the square, there will be certain rectangles, with one side γ, and the other γ', where $\gamma \leq \gamma' < 2\gamma$. We divide each of the squares into which the T square has been subdivided into two triangles by the diagonal with slope -1, and the rectangles we divide by the diagonal with negative slope. At the vertices of these triangles we define a function $\phi_T(x, y)$ equal to $f_T(x, y)$, or, except at the top and right hand side of the T square, to $f(x, y)$ which has the same values. On the sides of the triangles we define the function by linear interpolation, and similarly inside each triangle by linear interpolation, *i.e.*, so that its graph over any one triangle is plane. We note that the function $\phi_T(x, y)$ possesses partial derivatives at all points except those on the sides of the triangles, and at these points there are only a finite number of values approached, namely two at the sides, and six at the vertices. These partial derivatives are all bounded, being in absolute value less than η/γ on the triangles not having a vertex on the top or right hand side of the T square, and in these triangles less than $2G/\gamma'$. Hence they are always in absolute value less than $2G/\gamma$, which is independent of T. Outside the square we define $\phi_T(x, y)$ by requiring it to be periodic in x and y with period T in each case.

Now let the Fourier coefficients, in the classical sense, as defined here on p. 212, for $f_T(x, y)$ be a_{mn} and for $\phi_T(x, y)$ be b_{mn}. Since they are obtained by linear processes, for $f_T(x, y) - \phi_T(x, y)$ they

will be $a_{mn}-b_{mn}$, and by applying Parseval's theorem, we find:

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}|a_{mn}-b_{mn}|^2=1/T^2\int_0^T\int_0^T|f_T(x,y)-\phi_T(x,y)|^2\,dx\,dy.$$

We may break up the region of integration, the T square, into a square of side $(N-1)\gamma$, with one corner at the origin, where N is the biggest integer in T/γ, and a remainder area, in magnitude $T^2-(N-1)^2\gamma^2=[T+(N-1)\gamma]\ [T-(N-1)\gamma]$ less than $4T\gamma$. But, by the choice of γ, over the square the integrand is less than $4\eta^2$, and over the remainder area it is at most $4G^2$. Hence the right member of our equation is less than:

$$\frac{(N-1)^2\gamma^2 4\eta^2}{T^2}+\frac{4T\gamma 4G^2}{T^2}<4\eta^2+\frac{16G^2\gamma}{T}.$$

This can be made less than $\delta/2$, by taking $\eta<\frac{1}{4}\sqrt{\delta}$, and then when γ is determined from it, $T>T_0=64G^2\gamma/\delta$.

But, since the function $\phi_T(x,y)$ has for its partial derivatives doubly periodic functions summable and with summable square, we may apply Parseval's theorem to these derivatives. The coefficients for $\partial\phi_T/\partial x$ are:

$$b_{mn}'=1/T^2\int_0^T\int_0^T\partial\phi_T/\partial x\;e^{-imKx-inKy}dy\,dx$$

$$=1/T^2\int_0^T\left[\phi_T(x,y)e^{-imKx-inKy}\right]_{x=0}^{x=T}dy$$

$$+imK/T^2\int_0^T\int_0^T\phi_T\,e^{-imKx-inKy}dy\,dx,$$

on integrating by parts. But, because of the continuity and periodicity of $\phi_T(x,y)$ in x, the integrated part vanishes, and we have:

$$b_{mn}'=imK\,b_{mn}.$$

Parseval's theorem, applied to $\partial\phi_T/\partial x$, gives:

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}m^2K^2|b_{mn}|^2=\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}|b_{mn}'|^2=1/T^2\int_0^T\int_0^T|\partial\phi_T/\partial x|^2dxdy\leq 4G^2/\gamma^2$$

where the inequality follows from the upper limit on $|\partial\phi_T/\partial x|$. If now we take:

$$\Omega>\frac{4G}{\gamma\sqrt{\delta}},$$

we shall have:

$$\sum_{|m|>\Omega/K}\sum_{n=-\infty}^{\infty}|b_{mn}|^2<4G^2/\gamma^2\Omega^2<\delta/4.$$

By proceeding in an entirely similar manner with $\partial\phi_T/\partial y$, we may show that:

$$\sum_{m=-\infty}^{\infty}\sum_{|n|>\Omega/K}|b_{mn}|^2<\delta/4.$$

From these two inequalities, we find:

$$\sum_{|m|>\Omega/K \text{ or } |n|>\Omega/K}|b_{mn}|^2<\delta/2.$$

But we have already shown that:

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}|a_{mn}-b_{mn}|^2<\delta/2.$$

We combine these two inequalities by noting that, for any two real numbers A and B, we have:

$$A^2\leq A^2+(2B-A)^2=2[B^2+(A-B)^2].$$

Consequently:

$$\begin{aligned}&\sum_{|m|>\Omega/K \text{ or } |n|>\Omega/K}|a_{mn}|^2\\ &\quad\leq 2\Big(\sum_{|m|>\Omega/K \text{ or } |n|>\Omega/K}|b_{mn}|^2+\sum_{|m|>\Omega/K \text{ or } |n|>\Omega/K}|a_{mn}-b_{mn}|^2\Big)\\ &\quad\leq 2\Big(\sum_{|m|>\Omega/K \text{ or } |n|>\Omega/K}|b_{mn}|^2+\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}|a_{mn}-b_{mn}|^2\Big)\\ &\quad<\delta/2+\delta/2=\delta.\end{aligned}$$

This proves Theorem XII.

Theorem XIII. *Let λ_0, μ_0 be a pair of exponents yielding a vanishing coefficient for $f(x,y)$, i.e., $\alpha(\lambda_0,\mu_0)=M\{f(x,y)e^{-i\lambda_0 x-i\lambda_0 y}\}=0$. Then, for every $\delta>0$, there exists an $\omega>0$, and a $T_0>0$, such that, for $T>T_0$, we have:*

$$\sum_{|Km-\lambda_0|<\omega}\sum_{|Kn-\mu_0|<\omega}|a_{mn}|^2<\delta. \quad (K=2\pi/T).$$

(*a*) We shall first prove the theorem for the case in which $\lambda_0=0$, $\mu_0=0$. In this case, by our hypothesis, we have:

$$\lim_{T=\infty} 1/T^2 \int_0^T \int^T f(x, y)\ dx\ dy = a(0, 0) = 0.$$

We note that, by Theorem VII, Elementary Theory, p. 47,

$$\lim_{T=\infty} 1/T^2 \int_x^{x+T} \int_y^{y+T} f(x, y)\ dx\ dy = 0,$$

since it equals the limit just written, and, further, that the limit is approached uniformly with respect to x and y. Consequently, if we choose c large enough, and put

$$F(x, y) = 1/c^2 \int_x^{x+c} \int_y^{y+c} f(x, y)\ dx\ dy,$$

we shall have:

$$|F(x, y)|^2 < \delta/8.$$

Now let us put

$$\phi(x, y) = 1/c^2 \int_x^{x+c} \int_y^{y+c} f_T(x, y)\ dx\ dy. \quad (T>c)$$

This is clearly a doubly periodic function, with both periods equal to T, since $f_T(x, y)$ is of this character.

For its Fourier coefficients, we have:

$$b_{mn} = 1/T^2 \int_0^T \int_0^T \phi\ (x, y) e^{-im Kx - in Ky} dx\ dy$$

$$= 1/c^2 T^2 \int_0^T dx \int_0^T dy \int_x^{x+c} dx' \int_y^{y+c} dy'\ f_T(x', y') e^{-im Kx - in Ky}$$

$$= 1/c^2 T^2 \int_0^T dx' \int_0^T dy' \int_{x'}^{x'-c} dx \int_{y'}^{y'-c} dy\ f_T(x', y') e^{-im Kx - in Ky},$$

where, in finding the new limits we have made use of the fact that for a function periodic in x and x', both periods being $T>c$, we have:

$$\int_{T-c}^{T} dx \int_T^{x+c} dx' = \int_{-c}^{0} dx \int_0^{x+c} dx',$$

and of the corresponding formula in y and y'. The last expression gives

$$b_{mn} = 1/T^2 \int_0^T dx' \int^T dy' \, \frac{e^{imKc}-1}{imKc} \quad \frac{e^{inKc}-1}{inKc} f_T(x', y') e^{-imKx'-inKy'}$$

$$= a_{mn} \frac{e^{imKc}-1}{imKc} \quad \frac{e^{inKc}-1}{inKc} .$$

For m or n equal to 0, we must replace the indeterminate expression by its limiting value, unity, to make the last equation valid.

On applying Parseval's theorem to $\phi(x, y)$, we find:

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} |b_{mn}|^2 = 1/T^2 \int_0^T \int_0^T |\phi(x, y)|^2 \, dx \, dy.$$

But, since $f_T(x, y) = f(x, y)$ for $0 < x < T$, $0 < y < T$, we shall have $\phi(x, y) = F(x, y)$ for $0 < x < T-c$, $0 < y < T-c$. We now break up the integral on the right into two integrals, one over the square $0 < x < T-c$, $0 < y < T-c$, and the other over a remainder area, in magnitude $T^2 - (T-c)^2 = c(2T-c) < 2cT$. From the relation to $F(x, y)$ just noted, over the square the integrand is less than $\delta/8$, while over the remainder area it is less than G^2, G an upper bound for $|f(x, y)|$, since it is an average of $|f(x, y)|^2$ over a square of side c. Hence the integral is less than:

$$\frac{(T-c)^2}{T^2} \quad \frac{\delta}{8} + \frac{2cTG^2}{T^2} \quad < \quad \frac{\delta}{8} + \frac{2cG^2}{T} .$$

If we take $T > T_0 > 16\, cG^2/\delta$, this will be less than $\delta/4$, giving:

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} |a_{mn}|^2 \left| \frac{e^{imKc}-1}{imKc} \right|^2 \left| \frac{e^{inKc}-1}{inKc} \right|^2 = \sum_{1}^{\infty} \sum_{1}^{\infty} |b_{mn}|^2 < \delta/4.$$

Since the factors of $|a_{mn}|^2$ here approach unity when m and n approach zero K being fixed, and since for either m or n zero, the corresponding factor is to be replaced by unity, there will be an $\omega > 0$, such that, for $mK < \omega$, $nK < \omega$, the factors are both greater than $1/2$. Consequently we have:

$$\tfrac{1}{4} \sum_{|m|<\omega/K} \sum_{|n|<\omega/K} |a_{mn}|^2 < \sum_{|m|>\omega/K,\ |n|>\omega/K} \left| \frac{e^{imKc}-1}{imKc} \right|^2 \left| \frac{e^{inKc}-1}{inKc} \right|^2 < \delta/4.$$

Hence we have:

$$\sum_{|m|<\omega/K,\,|n|<\omega/K} |a_{mn}|^2<\delta,$$

which proves the theorem for the special case we were considering.

(*b*) We next consider the general case, where λ_0 and μ_0 are not necessarily both zero. We begin by writing

$$g(x, y)=f(x, y)e^{-i\lambda_0 x-i\mu_0 y}.$$

We note that, by our hypothesis on λ_0 and μ_0, we have:

$$M\{g(x, y)\}=M\{f(x, y)e^{-i\lambda_0 x-i\mu_0 y}\}=a(\lambda_0, \mu_0)=0.$$

Let $g_T(x, y)$ be the doubly periodic function with both periods equal to T which equals $g(x, y)$ for $0\leq x<T$, $0\leq y<T$, and let its Fourier coefficients be b_{mn}. Let $E_T(x, y)$ be the doubly periodic function with both periods equal to T which equals $e^{i\lambda_0 x+i\mu_0 y}$ for $0\leq x<T$, $0\leq y<T$, and let its Fourier coefficients be c_{mn}. On computing these c_{mn}, we find:

$$c_{mn}=1/T^2\int_0^T\int_0^T e^{i\lambda_0 x+i\mu_0 y}e^{-im Kx-in Ky}dx\,dy=c_m'c_n'',$$

where:

$$c_m'=1/T\int_0^T e^{i\lambda_0 x-imKx}\,dx=\frac{e^{i(\lambda_0-mK)T}-1}{i(\lambda_0-mK)T},$$

and:

$$c_n''=1/T\int_0^T e^{i\mu_0 y-inKy}\,dy=\frac{e^{i(\mu_0-nK)T}-1}{i(\mu_0-nK)T}.$$

If $\lambda_0=mK$, c_m' is to be replaced by unity, and similarly for c_n'' if $\mu_0=nK$. If $\lambda_0\neq mK$, we have:

$$|c_m'|\leq\frac{2}{|\lambda_0-mK|T}=\frac{2/KT}{|\lambda_0/K-m|}<\frac{1/2}{|\lambda_0/K-m|},$$

since $2/KT=1/\pi<1/2$. Now if an integer m_0 is chosen so that

$$m_0\leq\lambda_0/K<m_0+1,$$

we shall have:

$$\sum_{|m-m_0|>N}|c_m'|^2<\sum_{N-1}^{\infty}1/s^2.$$

As the series, $\sum_1^\infty 1/s^2$ is convergent, we may find an N_0 such that the remainder after N_0-2 terms is less than $\sqrt{\delta}/2G$, G being an upper bound for $|f(x, y)|$. Thus we shall have:

$$\sum_{|m-m_0|>N} |c_m'|^2 < \sqrt{\delta}/2G, \quad N > N_0.$$

In a similar manner, n_0 being an integer chosen so that $n_0 \leq \mu_0/K < n_0+1$, we may show that:

$$\sum_{|n-n_0|>N} |c_n''|^2 < \sqrt{\delta}/2G, \quad N > N_0.$$

Consequently we have:

$$\sum_{|n-n_0|>N} \sum_{|m-m_0|>N} |c_{mn}|^2 = \sum_{|n-n_0|>N} \sum_{|m-m_0|>N} c_m'^2 c_n''^2$$

$$\leq \sum_{|n-n_0|>N} c_m'^2 \sum_{|m-m_0|>N} c_n''^2 < \delta/4G^2.$$

Having selected N_0, we break up the Fourier series for $E_T(x, y)$ into two parts, putting

$$H(x, y) = \sum_{m_0-N}^{m_0+N} \sum_{n_0-N}^{n_0+n} c_{mn} e^{imKx+inKy}, \quad N > N_0$$

and

$R(x, y) = E_T(x, y) - H(x, y)$, corresponding to the series:

$$\sum_{|m-m_0|>N} \sum_{|n-n_0|>N} c_{mn} e^{imKx+inKy}.$$

We note that, from the application of Parseval's theorem to $E_T(x, y)$, we have:

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} |c_{mn}|^2 = 1/T^2 \int_0^T \int_0^T |E_T(x, y)|^2 dx\, dy = 1,$$

and hence

$$\sum_{m_0-N}^{m_0+N} \sum_{n_0-N}^{n_0+N} |c_{mn}|^2 \leq 1.$$

Also, from the application of Parseval's theorem to $R(x, y)$, we have:

$$1/T^2\int_0^T\int_0^T |R(x, y)|^2\,dx\,dy = \sum_{|m-m_0|>N}\sum_{|n-n_0|>N} |c_{mn}|^2 < \delta/4G^2.$$

We now recall that:

$$f_T(x, y) = g_T(x,y)\ E_T(x, y) = g_T(x, y)\,H(x, y) + g_T(x, y)R(x, y).$$

Writing a_{mn}' for the Fourier coefficients of $g_T\ (x, y)\ H(x, y)$, and a_{mn}'' for those of $g_T(x, y)\ R(x, y)$, we have from the linear character of the coefficients:

$$a_{mn} = a_{mn}' + a_{mn}''.$$

On applying Parseval's theorem to $g_T(x, y)\ R(x, y)$, we find:

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} |a_{mn}''|^2 = 1/T^2\int_0^T\int_0^T |g_T(x, y)\ R(x, y)|^2 dx\,dy < G^2 \cdot \delta/4G^2 = \delta/4.$$

The inequality follows from the mean value theorem, and the fact that G, being an upper bound for $|f(x, y)|$, is also one for $|g_T(x, y)|$ which is equal to it, since the factor by which the functions differ is unity in absolute value.

The series for $g_T(x, y)$ is:

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} b_{mn}\, e^{-imKx - inKy},$$

while that for $H(x, y)$ is:

$$\sum_{m_0-N}^{m_0+N}\sum_{n_0-N}^{n_0+N} c_{mn}\, e^{-imKx - inKy}.$$

Since the latter contains only a finite number of terms, we may find the series for the product of these functions, $g_T(x, y)\ H(x, y)$ by formal multiplication. We thus find:

$$|a_{mn}'| = \left|\sum_{m_0-N}^{m_0+N}\sum_{n_0-N}^{n_0+N} c_{pq} b_{m-p,\,n-q}\right| \leq \sum_{m_0-N}^{m_0+N}\sum_{n_0-N}^{n_0+N} |c_{pq}| \cdot |b_{m-p,\,n-q}|.$$

We now apply the Schwartzian inequality,

$$(\Sigma L_i M_i)^2 \leq (\Sigma L_i^2)(\Sigma M_i^2),$$

obtaining:

$$|a_{mn}'|^2 \leq \sum_{m_0-N}^{m_0+N}\sum_{n_0-N}^{n_0+N}|c_{pq}|^2 \sum_{m_0-N}^{m_0+N}\sum_{n_0+N}^{n_0+N}|b_{m-p,\,n-q}|^2$$

$$\leq \sum_{m_0-N}^{m_0+N}\sum_{n_0-N}^{n_0+N}|b_{m-p,\,n-q}|^2.$$

We now apply case (a) of this theorem, which has already been proved, to the function $g(x, y)$ for which 0, 0 is a pair of exponents yielding a vanishing coefficient. We may find an $\omega>0$, and a $T_1(\delta)$, such that, for $T>T_1$, we have:

$$\sum_{|m|<2\omega/K}\sum_{|n|<2\omega/K}|b_{mn}|^2<\frac{\delta}{4(2N+1)^2}.$$

We now set $T_0>T_1$, $2\pi(N+1)/\omega$. The last condition insures that, when $T>T_0$, $|Km-\lambda_0|<\omega$, $|Kn-\mu_0|<\omega$, the values of the subscripts in the sum which gives a limit on $|a_{mn}'|^2$, namely $m-p$, $n-q$ are less than $2\omega/K$. For, when $m_0-N\leq p\leq m_0+N$, we have:

$$|pK-\lambda_0|=K|p-\lambda_0/K|\leq K(N+1)=2\pi(N+1)/T\leq\omega.$$

Consequently, if $|Km-\lambda_0|<w$, we will have:

$$|m-p|=1/K\,|Km-Kp|\leq 1/K[\,|Km-\lambda_0|+|Kp-\lambda_0|]<2\omega/K.$$

Similarly, if $|Kn-\mu_0|<\omega$, we will have $|n-q|<2\omega/K$.

From the inequality derived above for $|a_{mn}'|^2$, we find:

$$\sum_{|Km-\lambda_0|<\omega}\sum_{|Kn-\mu_0|<\omega}|a_{mn}'|^2\leq\sum_{|Km-\lambda_0|<\omega}\sum_{|Kn-\mu_0|<\omega}\sum_{m_0-N}^{m_0+N}\sum_{n_0-N}^{n_0+N}|b_{m-p,\,n-q}|^2.$$

Since p and q each take on at most $2N+1$ values, there are at most $(2N+1)^2$ pairs of values taken on by p, q. Hence a fixed pair of subscripts on b will appear at most $(2N+1)^2$ times, and we see that:

$$\sum_{|Km-\lambda_0|<\omega}\sum_{|Kn-\mu_0|<\omega}|a_{mn}'|^2\leq(2N+1)^2\sum_{|Km-\lambda_0|<\omega}\sum_{|Kn-\mu_0|<\omega}|b_{m-p,\,n-q}|^2$$

$$\leq(2N+1)^2\sum_{|m-p|<2\omega/K}\sum_{|n-q|<2\omega/K}|b_{m-p,\,n-q}|^2<\delta/4.$$

From the inequality:

$$|a_{mn}|^2=|a_{mn}'+a_{mn}''|^2\leq 2\{|a_{mn}'|^2+|a_{mn}''|^2\},$$

we obtain:

$$\sum_{|Km-\lambda_p|<\omega}\sum_{|Kn-\mu_q|<\omega}|a_{mn}|^2\leq \sum_{|Km-\lambda_p|<\omega}\sum_{|Kn-\mu_q|<\omega}2\{|a_{mn}'|^2+|a_{mn}''|^2\},$$

$$\leq 2\sum_{|Km-\lambda_p|<\omega}\sum_{|Kn-\mu_q|<\omega}|a_{mn}'|^2+2\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}|a_{mn}''|^2$$

$$\leq 2(\delta/4)+2(\delta/4)=\delta,$$

which is the relation we were to prove.

Theorem XIV. *Let λ_p, μ_q be a pair of exponents yielding a non-vanishing coefficient for $f(x, y)$, i.e., $a(\lambda_p, \mu_q)=M\{f(x, y)e^{-i\lambda_p x-i\mu_q y}\}=A_{pq}\neq 0$. Then, for every $\delta>0$, there exists an $\omega>0$, and a $T_0>0$, such that, for $T>T_0$, we have:*

$$\sum_{|Km-\lambda_p|<\omega}\sum_{|Kn-\mu_q|<\omega}|a_{mn}|^2<|A_{pq}|^2+\delta.$$

We shall prove this theorem by means of the previous one. To do this, we put

$$f(x, y)=A_{pq}e^{i\lambda_p x+i\mu_q y}+g(x, y).$$

Let $g_T(x, y)$ be the doubly periodic function with both periods equal to T which equals $g(x, y)$ for $0\leq x<T$, $0\leq y<T$, and let its Fourier coefficients be b_{mn}. Let $E_T(x, y)$ be the doubly periodic function with both periods equal to T which equals $A_{pq}e^{i\lambda_p x+i\mu_q y}$ for $0\leq x<T$, $0\leq y<T$, and let its Fourier coefficients be c_{mn}. From the linear character of these coefficients, we have:

$$a_{mn}=c_{mn}+b_{mn}.$$

Also, since the operation of taking the mean is linear, we have:

$$M\{g(x, y)e^{-i\lambda_p x-i\mu_q y}\}=M\{f(x, y)e^{-i\lambda_p x-i\mu_q y}\}-M\{A_{pq}\}$$
$$=A_{pq}-A_{pq}=0.$$

Consequently we may apply Theorem XIII to $g(x, y)$, and find an $\omega>0$, and a $T_0>0$, such that, for $T>T_0$;

$$\sum_{|Km-\lambda_p|<\omega}\sum_{|Kn-\mu_q|<\omega}|b_{mn}|^2<\frac{\delta^2}{2\delta+4|A_{pq}|^2}.$$

By applying Parseval's theorem to $E_T(x, y)$ we find:

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} |c_{mn}|^2 = 1/T^2 \int_0^T \int_0^T E_T(x, y)dx\, dy = |A_{pq}|^2,$$

and hence:

$$\sum_{|Km-\lambda_p|<\omega} \sum_{|Kn-\mu_q|<\omega} |c_{mn}|^2 \leq |A_{pq}|^2.$$

We combine these results by using the inequality:

$$|L+M|^2 \leq (1+e^2)|L|^2 + (1+1/e^2)|M|^2,$$

which easily follows from $(e|L| - |M|/e)^2 \geq 0$.

On putting $e = |A_{pq}|\sqrt{2/\delta}$, we find from it:

$$|a_{mn}|^2 = |b_{mn}+c_{mn}|^2 \leq \left(1+\frac{2|A_{pq}|^2}{\delta}\right)|b_{mn}|^2 + \left(1+\frac{\delta}{2|A_{pq}|^2}\right)|c_{mn}|^2.$$

From this and the inequalities given above, we see that:

$$\begin{aligned}\sum_{|Km-\lambda_p|<\omega} \sum_{|Kn-\mu_q|<\omega} |a_{mn}|^2 &\leq \left(1+\frac{2|A_{pq}|^2}{\delta}\right)\frac{\delta^2}{2\delta+4|A_{pq}|^2} \\ &\quad + \left(1+\frac{\delta}{2|A_{pq}|^2}\right)|A_{pq}|^2, \\ &\leq \delta/2 + |A_{pq}|^2 + \delta/2 = |A_{pq}|^2 + \delta.\end{aligned}$$

This establishes the theorem.

5. The fundamental theorem.

The last four theorems give as an immediate consequence the fundamental theorem toward which we have been working.

Theorem XV. *Let A_{mn} denote the non-vanishing coefficients of the almost periodic function, $f(x, y)$. Then we have:*

$$\sum_{1}^{\infty}\sum_{1}^{\infty} |A_{mn}|^2 = M\{f(x, y)\}^2.$$

We first use Theorem XII to find an $\Omega > 0$, and a $T' > 0$, such that, for $T > T'$, we have:

$$\sum_{|m|>\Omega/K} \sum_{|n|>\Omega/K} |a_{mn}|^2 < \delta/4.$$

Next we take this Ω, and apply Theorem XI to find a finite number, M, of points $P_i=(\lambda_i, \mu_i)$ in the square $|\lambda|<\Omega$, $|\mu|<\Omega$, such that for any $\omega>0$, for all sufficiently great T, $T>T_0(\omega)$, we shall have

$$\Sigma'' \; \Sigma'' \; |a_{mn}|^2<\delta/4,$$

the double primes meaning that the summation extends over all values of m and n for which $\lambda=mK$, $\mu=nK$, $K=2\pi/T$, are inside the square $|mK|<\Omega$, $|nK|<\Omega$, and outside the M squares $|mK-\lambda_i|<\omega$, $|nK-\mu_i|<\omega$.

To fix on ω, for each point $P_i=(\lambda_i, \mu_i)$ for which $(\lambda_i, \mu_i)=0$, we apply Theorem XIII to find an $\omega_i>0$, and a $T_i>0$ such that for $T>T_i$, we have:

$$\sum_{|Km-\lambda_i|<\omega_i} \sum_{|Kn-\mu_i|<\omega_i} |a_{mn}|^2<\delta/2M. \quad (\lambda_i, \mu_i)=0.$$

We also apply Theorem XIV to find an $\omega_j>0$, and a $T_j>0$ for each point $P_j=(\lambda_j, \mu_j)$ for which $(\lambda_j, \mu_j)=A_{pq}\neq 0$, such that, for $T>T_j$, we have:

$$\sum_{|Km-\lambda_j|<\omega_j|} \sum_{|n-\mu_j|<\omega_j} |a_{mn}|^2<|A_{pq}|^2+\delta/2M.$$

We take the ω to be used in the second part of Theorem XI less than all of the ω_i, ω_j, and find $T_0(\omega)$. If now we take $T(\delta)$ greater than T', $T_0(\omega)$, and all of the T_i, T_j, all our inequalities will hold simultaneously, and we shall have:

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} |a_{mn}|^2<\sum_{1}^{\infty} \sum_{1}^{\infty} |A_{mn}|^2+\delta/4+\delta/4+M(\delta/2M)=\sum_{1}^{\infty} \sum_{1}^{\infty} |A_{mn}|^2+\delta.$$

But, since, on applying Parseval's theorem to $f_T(x, y)$ we find:

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} |a_{mn}|^2=\frac{1}{T^2}\int_0^T \int_0^T |f_T(x, y)|^2 \, dx \, dy,$$

and from Theorem XIII, Elementary Theory, p. 54, we have:

$$\sum_{1}^{\infty} \sum_{1}^{\infty} |A_{mn}|^2 \leq M\{|f(x, y)|^2\},$$

we must have:

$$1/T^2\int_0^T\int_0^T |f(x, y)|^2\,dx\,dy-\delta<\sum_1^\infty\sum_1^\infty |A_{mn}|^2\leq M\{\,|f(x, y)|^2\}.$$

The T in this relation depends on δ, in as much as it must be chosen $>T(\delta)$. Now give to δ a sequence of values approaching zero, *e.g.*, $1/n$ and to T a sequence of values becoming infinite, *e.g.*, $>T(1/n), >n$. The left member of the above relation approaches $M\{\,|f(x, y)|^2\}$, while the other terms stay fixed, so that we have:

$$M\{\,|f(x, y)|^2\}\leq\sum_1^\infty\sum_1^\infty |A_{mn}|^2\leq M\{\,|f(x, y)|^2\},$$

or:

$$\sum_1^\infty\sum_1^\infty |A_{mn}|^2=M\{\,|f(x, y)|^2\},$$

which forms the content of the fundamental theorem.

We may state the fundamental theorem in various alternative forms. We give some of these in the following two theorems.

Theorem XVI. *Let A_{mn} denote the non-vanishing coefficients of the almost periodic function $f(x, y)$. Then we have:*

$$\lim_{N=\infty,\,M=\infty} M\{\,|f(x, y)-\sum_1^M\sum_1^N A_{mn}e^{i\lambda_m x+i\mu_n y}|^2\}=0.$$

This follows at once from Theorem XV, and Theorem IX, Elementary Theory, p. 51. For, on replacing $\alpha(\lambda_m, \mu_n)$ by A_{mn} in the statement of the latter theorem, we have:

$$M\{\,|f(x, y)-\sum_1^M\sum_1^N A_{mn}e^{i\lambda_m x+i\mu_n y}|^2\}=M\{\,|f(x, y)|^2\}-\sum_1^M\sum_1^N |A_{mn}|^2,$$

and by Theorem XV, the limit of the right-hand side of this equation as M and N become infinite is zero.

Corollary 1. *For any $\epsilon>0$, and $\delta>0$, we may find a pair $M_0>0$, $N_0>0$, such that, for $M>M_0$, $N>N_0$, the proportion of area throughout the plane for which the absolute value of the error of the generalized Fourier series to MN terms, i.e.:*

$$\left|f(x, y)-\sum_1^M\sum_1^N A_{mn}e^{i\lambda_m x+i\mu_n y}\right|$$

is greater than ϵ, *is less than* δ. *Here, by proportion of area, we* mean $M\{E(x, y)\}$, $E(x, y)$ *being one when the error is greater than* ϵ, *and zero otherwise.*

For, consider the quantity:

$$M\left\{\left|f(x, y)-\sum_{1}^{M}\sum_{1}^{N} A_{mn}e^{i\lambda_m x+i\mu_n y}\right|^2\right\}.$$

By the theorem, we may find an M_0 and an N_0 such that, for $M>M_0$, $N>N_0$, this quantity is less than $\epsilon^2\delta$. But, from the definition of $E(x, y)$, the square error is greater than $\epsilon^2E(x, y)$, and hence the quantity we have just written is greater than $M\{\epsilon^2E(x, y)\}$. Thus we shall have, for $M>M_0$, $N>N_0$:

$$\epsilon^2M\{E(x, y)\}=M\{\epsilon^2E(x, y)\}<\epsilon^2\delta,$$

and hence

$$M\{E(x, y)\}<\delta.$$

Corollary 2. *Given* $\epsilon>0$ *and* $\delta>0$, *we may find three quantities* $M_0>0$, $N_0>0$, $A_0>0$, *such that in any square of side* $A>A_0$ *the proportion of area for which:*

$$\left|f(x,y)-\sum_{1}^{M_0}\sum_{1}^{N_0} A_{mn}e^{i\lambda_m+i\mu_n}\right|>\epsilon,$$

is less than δ.

We first select M_0 and N_0 by the theorem so that:

$$M\left\{\left|f(x, y)-\sum_{1}^{M_0}\sum_{1}^{N_0} A_{mn}e^{i\lambda_m+i\mu_n}\right|\right\}<\epsilon^2\delta/64.$$

The function

$$g(x, y)=\left|f(x, y)-\sum_{1}^{M_0}\sum_{1}^{N_0} A_{mn}e^{i\lambda_m+i\mu_n}\right|$$

is almost periodic, by Corollaries 1 and 4, Theorem IV, Elementary Theory, p. 44. Let A_0 be a displacement side for it belonging to $\epsilon/2$. We assert that this is the A_0 of the theorem.

For, suppose there were some square of side $A>A_0$ in which we had $g(x, y)>\epsilon$, over an area greater than δA^2. We draw a square with sides parallel to the axes, of side $2A$ containing this

square in its interior. We draw a square of side $4A$, with sides parallel to the axes concentric with the first, and build up a network of squares of this size with this as one square. In each of these squares we draw the concentric square of side $2A$. Since $2A > A_0$, we may find a displacement vector for $g(x, y)$ belonging to $\epsilon/2$ which will map the center of our first square of side $2A$ on this square, and hence take the square of side A we started with into one entirely inside the square of side $4A$. As each point at which $g(x, y) > \epsilon$ in our original square of side A goes into a point in which $g(x, y) > \epsilon/2$, we see that out of each square of the network, the area at which $g(x, y) > \epsilon/2$ is greater than δA^2. By Theorem VII, Elementary Theory, p. 47, we may take the integral which gives the mean value of $g(x, y)^2$ over the squares of the network, which shows that:

$$M\{g(x, y)^2\} > \frac{\delta A^2}{(4A)^2}\left(\frac{\epsilon}{2}\right)^2 = \epsilon^2\delta/64$$

a contradiction to our hypothesis that the square of side A existed.

We have already often made use of the fact that the generalized Fourier series for the sum of two functions is the sum of their two series term by term, as follows from the linear character of the operation of taking the mean. The series for the product of two function may be obtained by formally multiplying the two series together, as we shall now show in:

Theorem XVII. *Let A_{mn} be the non-vanishing coefficients for $f(x, y)$, and B_{pq} those for $g(x, y)$. The non-vanishing coefficients for $F(x, y) = f(x, y)\ g(x, y)$, which has been shown to be almost periodic in Corollary 2, Theorem IV, Elementary Theory, p. 44, are:*

$$C_{st} = \alpha\ (\lambda_s, \mu_t) = \sum_{\lambda_m' + \lambda_p'' = \lambda_s} \sum_{\mu_n' + \mu_q'' = \mu_t} A_{mn}\, B_{pq},$$

where the series converges absolutely, if it contains an infinite number of terms.

Since, by definition, we have:

$$C_{st} = M\left\{f(x, y)g(x, y)e^{-i\lambda_s x - i\mu_t y}\right\},$$

we must show that:

$$M\{f(x, y)g(x, y)e^{-i\lambda_s-i\mu_t y}\} = \sum_{\lambda_m'+\lambda_p''=\lambda_s} \sum_{\mu_n'+\mu_q''=\mu_t} A_{mn}B_{pq}.$$

If we put $h(x, y)=g(x, y)e^{-i\lambda_s-i\mu_t}$, and letter its coefficients H_{uv} so that:

$$\lambda_u'''+\lambda_s=\lambda_p'', \quad \mu_v'''+\mu_t=\mu_q'',$$

we have:

$$H_{uv}=M\{h(x, y)e^{-i\lambda_u'''-i\mu_v'''}\}=M\{g(x, y)e^{-i\lambda_p''-i\mu_q''}\}=B_{pq}.$$

Consequently the relation we were to prove becomes:

$$M\{f(x, y)h(x, y)\} = \sum_{\lambda_m'+\lambda_u'''=0} \sum_{\mu_n'+\mu_v'''=0} A_{mn}H_{uv}.$$

If $h(x, y)$ is the conjugate function of $f(x, y)$, $\bar{f}(x, y)$, we have for its coefficients F_{uv}, assuming $\lambda_m'=-\lambda_u'''$, $\mu_n'=-\mu_v'''$

$$F_{uv}=M\{\bar{f}(x, y)e^{-i\lambda_u'''-i\mu_v'''}\}=\text{conj. } M\{f(x,y)e^{-i\lambda m'-i\mu n'}\}=\overline{A}_{mn}.$$

Thus for this choice of $h(x, y)$, the relation above becomes:

$$M\{f(x, y)\bar{f}(x, y)\}=\sum_1^\infty \sum_1^\infty A_{mn}\overline{A}_{mn},$$

which is true by Theorem XV, the series converging absolutely.

We go from this to the case in which $f(x, y)$ and $h(x, y)$ are any two *real* functions, by noting that in this case:

$$M\{f(x, y)\,h(x, y)\}=\tfrac{1}{4}M\{(f+h)^2-(f-h)^2\}$$

$$=\tfrac{1}{4}M\{(f+h)(\bar{f}+\bar{h})\}-\tfrac{1}{4}M\{(f-h)(\bar{f}-\bar{h})\}$$

$$=\tfrac{1}{4}\sum_1^\infty\sum_1^\infty(A_{mn}+H_{mn})(\overline{A}_{mn}+\overline{H}_{mn})-\tfrac{1}{4}\sum_1^\infty\sum_1^\infty(A_{mn}-H_{mn})(\overline{A}_{mn}-\overline{H}_{mn})$$

$$=\tfrac{1}{2}\sum_1^\infty\sum_1^\infty(A_{mn}\overline{H}_{mn}+\overline{A}_{mn}H_{mn})=\tfrac{1}{2}\sum_1^\infty\sum_1^\infty A_{mn}\overline{H}_{mn}+\tfrac{1}{2}\sum_1^\infty\sum_1^\infty\overline{A}_{uv}H_{uv}$$

$$=\sum_{\lambda_m=-\lambda_u}\sum_{\mu_n=-\mu_v}A_{mn}H_{uv}.$$

The series $\Sigma\Sigma A_{mn}\overline{H}_{mn}$ is here absolutely convergent, since

$$4|A_{mn}||H_{mn}| \leq |A_{mn}+H_{mn}|^2+|A_{mn}-H_{mn}|^2,$$

and thus our rearrangements of terms are valid.

We now proceed to the case of any two functions by a similar process. We write $f=f_1+if_2$, with A_{mn}' the coefficients of f_1, and A_{mn}'' those of f_2. Here f_1 and f_2 are real functions of x and y. Their coefficients are not, of course, real, but we have:

$$A_{mn}=A_{mn}'+iA_{mn}''.$$

Similarly we put $h=h_1+ih_2$, with H_{mn}' the coefficients of h_1, and H_{mn}'' those of h_2, so that:

$$H_{mn}=H_{mn}'+iH_{mn}''.$$

We now have:

$$M\{f(x,y)\,h(x,y)\}=M\{(f_1+if_2)\,(h_1+ih_2)\}$$

$$=M\{f_1h_1\}+M\{-f_2h_2\}+M\{if_1h_2\}+M\{if_2h_1\}$$

$$=\Sigma\Sigma A_{mn}'H_{uv}'-\Sigma\Sigma A_{mn}''H_{uv}''+i\Sigma\Sigma A_{mn}'H_{uv}''+i\Sigma\Sigma A_{mn}''H_{uv}'$$

$$=\Sigma\Sigma(A_{mn}'+iA_{mn}'')(H_{uv}'+iH_{uv}'')$$

$$=\sum_{\lambda_m=\lambda_u}\sum_{\mu_n=\mu_v}A_{mn}H_{uv}.$$

Here also our rearrangements of the terms are valid, since all the series we are dealing with are absolutely convergent. Thus we have proved the theorem.

We note that in proving this theorem, we have shown that the validity of the multiplication process is implied by, and implies the fundamental theorem.

6. The uniqueness theorem.

We shall now show that the generalized Fourier series of an almost periodic function uniquely characterizes the function. That is:

Theorem XVIII. *If two almost periodic functions of two variables have the same generalized Fourier series, they are equal.*

For suppose the functions are $f(x,y)$ and $g(x,y)$, and put

$$h(x,y)=f(x,y)-g(x,y)$$

The coefficients of $h(x, y)$, H_{mn} being the difference of those of $f(x, y)$ and $g(x, y)$, which are identical, are all zero. Hence by Theorem XV, we have:

$$M\{h(x, y)\}^2 = \sum_1^\infty \sum_1^\infty |H_{mn}|^2 = 0.$$

From this we easily prove that $h(x, y)$ is identically zero. For, if there were any point in the plane at which $h(x, y)$ were $k \neq 0$, from its continuity we could surround this by a square of side δ with sides parallel to the axes such that, in this square we would have $|h(x, y)| > |k|/2$. Now select a, greater than δ, and a displacement side for $h(x, y)$ belonging to $|k|/4$. Divide the plane by a square network, with sides parallel to the axis, and equal to $2a$, the point we began with being the center of one square. For each square of side $2a$, we may find a displacement vector for $h(x, y)$ belonging to $|k|/4$ which takes the point we started with into a point inside the concentric square of side a. Consequently it takes the square of side δ into an equal square entirely inside the square of side $2a$. In this square, we have $|h(x, y)| > |k|/4$. But, by Theorem VII, Elementary Theory, p. 47, we may take the integral which gives the mean value of $|h(x, y)|^2$ over the squares of the network, which shows that:

$$M\{|h(x, y)|^2\} > \frac{\delta^2}{4a^2} \frac{|k|^2}{16} > 0.$$

This contradicts the equations given above, and shows that $h(x, y)$ is identically zero, so that $f(x, y) = g(x, y)$.

Corollary 1. *If the generalized Fourier series for an almost periodic function converges uniformly it converges to this function.*

By Theorem XII, Elementary Theory, p. 53, if the series converges uniformly, it converges to an almost periodic function with the given series as its generalized Fourier series. By the theorem we have just proved, this almost periodic function is equal to the given function.

Theorem XIX. *If two almost periodic functions of two variables have the same values in any region such that for any given R_0, a circle of radius $R > R_0$ lying wholly in the region can be found, the two functions are equal.*

From the property of our region, we may find a series of squares, of side T_n, each with sides parallel to the axes and wholly inside our region, such that T_n becomes infinite with n. If $f(x, y)$ and $g(x, y)$ are the two given functions, and $h(x, y)$ is their difference, by Theorem VII, Elementary Theory, p. 47, we may compute

$$M\{|h(x, y)|^2\}$$

from its values in these squares. As it is zero there, its value is zero, and by an argument similar to that used in proving Theorem XVIII, we may show that it is zero throughout.

Corollary. If two almost periodic functions of two variables have the same value in any angular sector of the plane, they are equal.

A CORRECTION

By PHILIP FRANKLIN

In my article on almost periodic functions of two variables, No. 1, Vol. V, of this *Journal*, the proof of Theorem III, on p. 43, is inadequate, as was called to my attention by Prof. Harold Bohr. The following proof should be substituted:

Let δ be selected, according to the corollary of Theorem II, so that if s_1, t_1 is a displacement vector for the first function $f_1(x, y)$ belonging to $\epsilon/2$, and $|s_2-s_1|<\delta$, $|t_2-t_1|<\delta$, s_2, t_2 will be a displacement vector for ϵ. Let r_1 and r_2 be displacement radii belonging to $\epsilon/4$ for $f_1(x, y)$ and $f_2(x, y)$ respectively. Select a number $a>2r_1$, $2r_2$. This will be a displacement side for both functions belonging to $\epsilon/4$. Draw lines parallel to the axes to form a lattice of squares, a on a side, with the origin as one vertex. In each of these squares, Q_j, we may find a vector s_1^j, t_1^j belonging to $\epsilon/4$ for $f_1(x, y)$, and a vector s_2^j, t_2^j belonging to $\epsilon/4$ for $f_2(x, y)$ Now consider the set of vectors $S^j=s_2^j-s_1^j$, $T^j=t_2^j-t_1^j$, all drawn. so that they have one extremity at the origin. Their other extremities all lie in a square of side $2a$ with center at the origin. Draw lines parallel to the axes to divide this square into N^2 squares q_i each of side $2a/N<\delta$. Some of these squares will not contain any of the points S^j, T^j. For each q_i which contains any of these points, pick out the square Q_k nearest the origin with its S^k, T^k in q_i. Now take $A>a$ so large that a square with center at the origin

and side $2A$ contains all of the Q_k just selected. We shall show that in every square of side $4A$ with sides parallel to the axes, and hence in every circle of radius $2A\sqrt{2}$, there is at least one displacement vector belonging to ϵ for both functions.

For, consider any such square of side $4A$. In the square of side $2A$ with the same center and parallel sides there is contained one of the squares of side a, Q_j. Find the S^j, T^j for this square, and note the square q_i in which it lies. There is a Q_k with its S^k, T^k in this same q_i. We may take $s=s_2^j-s_2^k$, $t=t_2^j-t_2^k$, as the common displacement vector sought. It is clearly inside the square of side $4A$, since s_2^j and t_2^j are each within A of the coördinates of the center of the square, while s_2^k and t_2^k are each numerically less than A. As it is the difference of two displacement vectors for $f_2(x, y)$ belonging to $\epsilon/4$, it is a displacement vector belonging to $\epsilon/2$, and a fortiori to ϵ, for $f_2(x, y)$. Finally we note that the vector $s'=s_1^j-s_1^k$, $t'=t_1^j-t_1^k$, being the difference of two displacement vectors for $f_1(x, y)$ belonging to $\epsilon/4$, is a displacement vector for $f_1(x, y)$ belonging to $\epsilon/2$. But, we have:

$$|s-s'| = |s_2^j-s_2^k-s_1^j+s_1^k| = |S^j-S^k| < \delta.$$
$$|t-t'| = |t_2^j-t_2^k-t_1^j+t_1^k| = |T^j-T^k| < \delta.$$

Consequently, from our choice of δ, the vector s, t is a displacement vector for $f_1(x, y)$ belonging to ϵ.

We note the following minor corrections:

P. 49, 4 lines from botton, the 0 and 1 should be interchanged in the formula, the result in words being correctly stated.

P. 52, line 17. The Fourier summation should run from $-\infty$ to ∞.

P. 53, line 3 and later, the series should run from $-M$, $-N$ to M, N.

P. 52, 2 lines from bottom and later, the exponent $imx+iny$ should be $imKx+inLy$.

P. 53, lines 6-7 should read: "We may do this since $f(x, y)$ may be looked on as a linear combination, with some imaginary coefficients, of real functions either even or odd in x, and either even or odd in y. Each of these functions, being periodic and continuous, is thus a real, continuous function of $X=e^{iKx}\pm e^{-iKx}$, $Y=e^{iLy}\pm e^{-iLy}$. Accordingly . . ."

COMPOUNDS OF SCOTT PRODUCT DETERMINANTS

By LEPINE HALL RICE

In a previous paper,[1] theorems were obtained concerning the adjoint of a determinant of higher class, and in particular concerning the adjoint of a Scott product. These should be but special cases of relations involving any compounds of such determinants.

1. **Compounds and introsigned compounds.** As in the case of a 2-way determinant, so for a determinant of any class, the mth compound has for elements the minors of order m of the original determinant; if this be of class p, of order n, and of any signancy, the compound will be of the same class and signancy and of order $\binom{n}{m}$. In each of the p directions the $\binom{n}{m}$ layers are understood to be arranged alike.

To every minor of a determinant a sign may be prefixed which we shall call its *displacement-sign*. The sign is determined by the number of interchanges of layers of signant directions (disregarding the number of interchanges of layers of nonsignant directions) necessary to move the minor to the leading position in the matrix, that is, to the region common to the first m layers of each direction, if the minor be of order m. The displacement-signs of two complementary minors are evidently the same; and either minor with the displacement-sign prefixed is called the *algebraic complement* or *cofactor* of the other minor.

If to every element of the mth compound the displacement-sign proper to it as a minor of the original determinant be prefixed, the compound will be said to be *introsigned*. Thus in particular the adjoint is the introsigned $(n-1)$th compound.

In this work a notation which displays the ranges of a determinant is useful.[2] Thus the general determinant may be written

$$A = \left| a \begin{array}{c} \overset{\pm}{\overbrace{}} \quad \overset{+}{\overbrace{\phantom{u_{f+1}\ u_p}}} \\ u_1\ u_2 \ldots u_f\ u_{f+1}\ u_p \\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots \\ n\ n\ n\ n\ n \end{array} \right|, \tag{1}$$

the signant indices being put first merely for convenience.

[1] *Adjoint and inverse determinants and matrices*, this Journal, vol. V, No. 1, December, 1925, p. 55. As to Scott products, etc., see L. H. Rice, *P-way determinants*, Amer. J. of Math. 40 (1918), p. 242.

[2] Cf. A. Cayley, *On the theory of determinants*, Trans. Cambridge Phil. Soc. vol. VIII (1843), p. 75; Coll. Math. Papers, vol. I, p. 63. See §2 of that paper.

If in each of the p ranges $n-m$ index-values be deleted, a minor M of order m is obtained. We may denote the retained values in the rth range by

$$i_{r|1}\ i_{r|2}\ \cdots\ i_{r|m} \tag{2}$$

and the deleted values by

$$j_{r|1}\ j_{r|2}\ \cdots\ j_{r|n-m}, \tag{3}$$

and write the minor as

$$M=\left|\ a\ \begin{matrix}\pm\\ i_{1|1}\ i_{2|1}\ \cdots\ i_{f|1}\\ \vdots\ \ \vdots\ \ \ \vdots\\ m\ \ m\ \ \ m\end{matrix}\ \begin{matrix}+\\ i_{f+1|1}\ \cdots\ i_{p|1}\\ \vdots\ \ \ \vdots\\ m\ \ \ m\end{matrix}\ \right|. \tag{4}$$

For this minor the displacement-sign, in Cayley's notation, is

$$\prod_{r=1}^{f}\left(\pm\ i_{r|1}\ i_{r|2}\ \cdots\ i_{r|m}\ j_{r|1}\ j_{r|2}\ \cdots\ j_{r|n-m}\right). \tag{5}$$

Theorem I. *The value of a compound determinant is not changed by introsigning it.*

For, the compound may be introsigned by multiplying by -1 each layer of the first direction for which

$$\pm_{i_{1|1}\ i_{1|2}\ \cdots\ i_{1|m}\ j_{1|1}\ j_{1|2}\ \cdots\ j_{1|n-m}}=-1, \tag{6}$$

with like treatment of the same layers of each signant direction. And there are necessarily an even number of signant directions.

Example. Starting with a 3-way determinant of order 3 in which the second index is nonsignant, and forming the introsigned 2d compound, with the layers of each direction arranged in the order of the pairs of index-values 23, 13, 12, leaving these *in loco* and remembering that in the signant ranges they contribute respectively the signs $+$, $-$, $+$ to the displacement-sign, explicitly indicating the signancy $\pm+\pm$ in every element, and displaying the layers of the second direction, we have:

$$\left|\begin{array}{ccc} +\left|a^{\pm+\pm}_{\cdot\,\cdot\,\cdot\,\cdot}\right| & -\left|a^{\pm+\pm}_{\cdot\,\cdot\,\cdot\,1}\right| & +\left|a^{\pm+\pm}_{\cdot\,\cdot\,\cdot\,1}\right| \\ -\left|a^{\pm+\pm}_{1\,\cdot\,\cdot\,\cdot}\right| & +\left|a^{\pm+\pm}_{1\,\cdot\,\cdot\,1}\right| & -\left|a^{\pm+\pm}_{1\,\cdot\,\cdot\,1}\right| \\ +\left|a^{\pm+\pm}_{1\,\cdot\,\cdot\,\cdot}\right| & -\left|a^{\pm+\pm}_{1\,\cdot\,\cdot\,1}\right| & +\left|a^{\pm+\pm}_{1\,\cdot\,\cdot\,1}\right| \end{array}\right| \left|\begin{array}{ccc} +\left|a^{\pm+\pm}_{\cdot\,1\,\cdot\,\cdot}\right| & -\left|a^{\pm+\pm}_{\cdot\,1\,\cdot\,1}\right| & +\left|a^{\pm+\pm}_{\cdot\,1\,\cdot\,1}\right| \\ -\left|a^{\pm+\pm}_{1\,1\,\cdot\,\cdot}\right| & +\left|a^{\pm+\pm}_{1\,1\,\cdot\,1}\right| & -\left|a^{\pm+\pm}_{1\,1\,\cdot\,1}\right| \\ +\left|a^{\pm+\pm}_{1\,1\,\cdot\,\cdot}\right| & -\left|a^{\pm+\pm}_{1\,1\,\cdot\,1}\right| & +\left|a^{\pm+\pm}_{1\,1\,\cdot\,1}\right| \end{array}\right| \left|\begin{array}{ccc} +\left|a^{\pm+\pm}_{\cdot\,1\,1\,\cdot}\right| & -\left|a^{\pm+\pm}_{\cdot\,1\,1\,1}\right| & +\left|a^{\pm+\pm}_{\cdot\,1\,1\,1}\right| \\ -\left|a^{\pm+\pm}_{1\,1\,1\,\cdot}\right| & +\left|a^{\pm+\pm}_{1\,1\,1\,1}\right| & -\left|a^{\pm+\pm}_{1\,1\,1\,1}\right| \\ +\left|a^{\pm+\pm}_{1\,1\,1\,\cdot}\right| & -\left|a^{\pm+\pm}_{1\,1\,1\,1}\right| & +\left|a^{\pm+\pm}_{1\,1\,1\,1}\right| \end{array}\right| \tag{7}$$

This, in fact, is the adjoint of the determinant from whose elements it is formed, since m here equals $n-1$. Going from $n=3$ to $n=4$ we shall no longer have the adjoint but a determinant of order 6, the structure of whose matrix will be sufficiently indicated by writing down its leading file of the third direction (corresponding to the upper left-hand line in (7)), *viz.*,

$$+\left|a^{\pm+\pm\pm}_{\cdot\,\cdot\,\cdot\,\cdot}\right|,\quad -\left|a^{\pm+\pm\pm}_{\cdot\,\cdot\,\cdot\,2}\right|,\quad +\left|a^{\pm+\pm\pm}_{\cdot\,\cdot\,2\,\cdot}\right|,\quad +\left|a^{\pm+\pm\pm}_{\cdot\,\cdot\,2\,2}\right|,\quad -\left|a^{\pm+\pm\pm}_{\cdot\,\cdot\,\cdot\,1}\right|,\quad +\left|a^{\pm+\pm\pm}_{\cdot\,\cdot\,1\,\cdot}\right|. \tag{8}$$

2. **Scott product of two determinants.** We start with two determinants A and B of classes p and q, of common order n, and of any signancies,

$$A=\left|a\ \frac{(\pm)_\theta}{u_1\ \ldots\ u_p}\right|, \qquad B=\left|b\ \frac{(\pm)_\phi}{v_1\ \ldots\ v_q}\right|$$

the symbols that denote the signancy being read " of the signancy θ " and " of the signancy ϕ". And we form a Scott product H of class $p+q-1$ whose first p indices correspond to the p indices of A and whose last q indices correspond to the q indices of B, corresponding indices taking the same signancy, so that the pth index of H takes both the signancy of the last index of A and the signancy of the first index of B; this mode of statement being logically equivalent but for our use preferable to the rule as commonly given. We may express all this by writing

$$H=\left|h\ \frac{(\pm)_\theta \qquad (\pm)_\phi}{u_1\ \ldots\ u_{p-1}\ u_p\ v_2\ \ldots\ v_n}\right| \tag{9}$$

the general element being given by the prescription

$$h_{u_1 \cdots u_{p-1} v_{p1} \cdots v_q} = a_{u_1 \cdots u_{p-1} v_{p1}} b_{v_{p1} v_1 \cdots v_q} \tag{10}$$

(7)

Let us explicitly state, in matrix language, what makes H a Scott product of A and B. Any element of H lies at the intersection of $p+q-1$ layers, one of each of the $p+q-1$ directions, and this element is the product of (*i*) that element of A which lies at the intersection of the p layers, one of each direction, that correspond, one to one, to the first p of the $p+q-1$ layers in H, and (*ii*) that element of B which lies at the intersection of the q layers, one of each direction, that correspond, one to one, to the last q of the $p+q-1$ layers in H. And the signancies of A and B are continued in H.

As we go on, indices become complicated, but if they denote the same layers of corresponding directions, in so far the condition for a Scott product is met.

Furthermore if either A or B or both be modified by multiplying a layer or group of layers of a signant direction by -1 and the same layer or group of layers of every other signant direction by -1 (which does not change the value of A or B) and H be modified by multiplying the same layer or group of layers of each corresponding direction by -1, H is still a Scott product of A and B.

To any minor of H there *correspond* a minor of A and a minor of B, namely, the minor of A (B) designated by the same p (q) sets of index-values that form the first p (last q) ranges of the minor of H. The following evident theorem was implied in some of the work of the previous paper:

Theorem II. *Any minor of a Scott product of two determinants is a Scott product of the corresponding minors of the determinants, and the displacement-sign of the product minor is the product of the displacement-signs of the factor minors.*

3. **Compound of Scott product.** Forming now the mth compound of H we see that each element is by Theorem II so constituted that we have directly

Theorem III. *The (introsigned or not introsigned) mth compound of a Scott product of two determinants is a like Scott product of the (introsigned or not introsigned) mth compounds of the factor determinants.*

Now Theorem II applies again to the compound determinants of Theorem III to give fresh sets of minors, of order m' say, forming the matrices of the m'th compounds of the mth compounds, and Theorem III applies to these; and the process may be repeated indefinitely.

Example. As an example, we shall take a 3-way determinant A and a 2-way determinant B of order 4, form a Scott product H, pass to the introsigned 2d compounds of A, B, and H, and exhibit a minor of the introsigned compound of H and the corresponding minors of the introsigned compounds of A and B, whose Scott product it will be seen to be. Let

$$A=\left|a^{\pm+\pm}_{\substack{1\,1\,1\\2\,2\,2\\3\,3\,3\\4\,4\,4}}\right| \quad \text{and} \quad B=\left|b^{\pm\pm}_{\substack{1\,1\\2\,2\\3\,3\\4\,4}}\right|, \quad \text{then} \quad H=\left|h^{\overset{a}{\pm+\pm}\;\overset{b}{\pm\pm}}_{\substack{1\,1\,1\,1\\2\,2\,2\,2\\3\,3\,3\,3\\4\,4\,4\,4}}\right|; \tag{11}$$

that is, the signancy of H is $\pm++\pm$. The 2d compounds of A, B, and H are of order 6. Take that minor of order 2 of the introsigned 2d compound of H in which the index-values are (1) 12, 13, (2) 14, 24, (3) 23, 24, (4) 13, 34, in the respective ranges; (1) and (4) contributing therefore to the introsign the respective sign-factors (1) $+$, $-$, (4) $-$, $+$; *viz.*,

$$\left|\begin{array}{cccc} -\left|h^{\pm++\pm}_{\substack{1\,1\,.\,1\\2\,.\,2\,.\\.\,.\,3\,3\\.\,4\,.\,.}}\right| & +\left|h^{\pm++\pm}_{\substack{1\,1\,.\,.\\2\,.\,2\,.\\.\,.\,3\,3\\.\,4\,.\,4}}\right| & -\left|h^{\pm++\pm}_{\substack{1\,.\,.\,1\\2\,2\,2\,.\\.\,.\,3\,3\\.\,4\,.\,.}}\right| & +\left|h^{\pm++\pm}_{\substack{1\,.\,.\,.\\2\,2\,2\,.\\.\,.\,3\,3\\.\,4\,.\,4}}\right| \\ -\left|h^{\pm++\pm}_{\substack{1\,1\,.\,1\\2\,.\,2\,.\\.\,.\,.\,3\\.\,4\,4\,.}}\right| & +\left|h^{\pm++\pm}_{\substack{1\,1\,.\,.\\2\,.\,2\,.\\.\,.\,.\,3\\.\,4\,4\,4}}\right| & -\left|h^{\pm++\pm}_{\substack{1\,.\,.\,1\\2\,2\,2\,.\\.\,.\,.\,3\\.\,4\,4\,.}}\right| & +\left|h^{\pm++\pm}_{\substack{1\,.\,.\,.\\2\,2\,2\,.\\.\,.\,.\,3\\.\,4\,4\,4}}\right| \\ \hline +\left|h^{\pm++\pm}_{\substack{1\,1\,.\,1\\.\,.\,2\,.\\3\,.\,3\,3\\.\,4\,.\,.}}\right| & -\left|h^{\pm++\pm}_{\substack{1\,1\,.\,.\\.\,.\,2\,.\\3\,.\,3\,3\\.\,4\,.\,4}}\right| & +\left|h^{\pm++\pm}_{\substack{1\,.\,.\,1\\.\,2\,2\,.\\3\,.\,3\,3\\.\,4\,.\,.}}\right| & -\left|h^{\pm++\pm}_{\substack{1\,.\,.\,.\\.\,2\,2\,.\\3\,.\,3\,3\\.\,4\,.\,4}}\right| \\ +\left|h^{\pm++\pm}_{\substack{1\,1\,.\,1\\.\,.\,2\,.\\3\,.\,.\,3\\.\,4\,4\,.}}\right| & -\left|h^{\pm++\pm}_{\substack{1\,1\,.\,.\\.\,.\,2\,.\\3\,.\,.\,3\\.\,4\,4\,4}}\right| & +\left|h^{\pm++\pm}_{\substack{1\,.\,.\,1\\.\,2\,2\,.\\3\,.\,.\,3\\.\,4\,4\,.}}\right| & -\left|h^{\pm++\pm}_{\substack{1\,.\,.\,.\\3\,2\,2\,.\\.\,.\,.\,3\\.\,4\,4\,4}}\right| \end{array}\right|. \tag{12}$$

This minor is obviously the Scott product of the corresponding minors of the introsigned 2d compounds of A and B, *viz.*,

$$\begin{vmatrix} +\left|a^{\pm\pm}_{1\,2\,\cdot\,4}\right| & -\left|a^{\pm\pm}_{1\,2\,\cdot\,4}\right| & -\left|a^{\pm\pm}_{1\,3\,\cdot\,4}\right| & +\left|a^{\pm\pm}_{1\,3\,\cdot\,4}\right| \\ +\left|a^{\pm\pm}_{1\,2\,\cdot\,4}\right| & -\left|a^{\pm\pm}_{1\,2\,\cdot\,4}\right| & -\left|a^{\pm\pm}_{1\,3\,\cdot\,4}\right| & +\left|a^{\pm\pm}_{1\,3\,\cdot\,4}\right| \end{vmatrix}, \quad \begin{vmatrix} -\left|b^{\pm\pm}_{2\,3\,\cdot}\right| & +\left|b^{\pm\pm}_{2\,3\,\cdot}\right| \\ +\left|b^{\pm\pm}_{2\,3\,\cdot}\right| & -\left|b^{\pm\pm}_{2\,3\,\cdot}\right| \end{vmatrix}. \tag{13}$$

4. **Scott product of more than two determinants.** A Scott product of any number of determinants may be formed by first forming a Scott product of two of them, then a Scott product of this and a third, and so on. At any stage, any index of the new factor may be linked with any index of the partial product, the signancy of that index in the new factor being superposed on its simple or complex signancy in the partial product.

For such a product the following extensions of Theorems II and III evidently hold.

Theorem IV. *Any minor of a Scott product of any number of determinants is a like Scott product of the corresponding minors of the determinants, and the displacement sign of the product minor is the product of the displacement-signs of the factor minors.*

Theorem V. *The (introsigned or not introsigned) mth compound of a Scott product of any number of determinants is a like Scott product of the (introsigned or not introsigned) mth compounds of the factor determinants.*

5. **Compound of Scott product of 2-way determinants.** In the theory of 2-way determinants the value of the mth compound Δ of a determinant A, and the value of any r-rowed minor Δ_r of Δ, are found in the following manner.[3] The introsigned $(n-m)$th compound Δ' is formed by replacing each element of Δ by its algebraic complement in A. The product of Δ and Δ' is formed by multiplying row into row, like-numbered rows giving

[3] *Theory of Determinants*, Scott and Mathews, Cambridge, 1904, Chap. VI, §§8, 9.

as each main-diagonal element a Laplace expansion of A and unlike rows giving zeros for the remaining elements, whence $\Delta\,\Delta' = A^{\binom{n}{m}}$, from which it easily follows that $\Delta = A^{\binom{n-1}{m-1}}$ and $\Delta' = A^{\binom{n-1}{m}}$. Returning to the product of Δ and Δ', Δ is now altered by replacing each of the $s = \binom{n}{m} - r$ rows that contribute no elements to Δ_r by

$$0, \ \ldots\ , 0, 1, 0, \ \ldots\ , 0,$$

the 1 standing, in the kth of the s rows, in the kth of the s columns that contribute no elements to Δ_r; the result being to substitute for Δ a determinant equal to $\pm\,\Delta_r$, the sign being the displacement-sign of Δ_r, and thereby to substitute $\pm\,\Delta_r\,\Delta'$ for $\Delta\,\Delta'$. The latter substitute has A's in the main diagonal only in the r rows corresponding to those of $\pm\,\Delta_r$ (with zeros elsewhere in these rows) while its remaining s rows are identical (individually and in their order) with the s columns of Δ' contributing to the complement Δ_s' of the r-rowed minor of Δ' corresponding (in position) to Δ_r in Δ; hence the substitute is equal to $A^r\,\Delta_s'$. Thus $\pm\,\Delta_r\,\Delta' = A^r\,\Delta_s'$, and since $\Delta' = A^{\binom{n-1}{m}}$ the final result is

$$\Delta_r = \pm\,\Delta_s' A^{r-\binom{n-1}{m}}, \tag{14}$$

or, any r-rowed minor of the mth compound of A is equal to the algebraic complement of the corresponding minor in the introsigned $(n-m)$th compound of A, multiplied by $A^{r-\binom{n-1}{m}}$. Since $r-\binom{n-1}{m}+s-\binom{n-1}{m-1}=0$, and since the displacement signs of complementary minors are the same, (14) may be put in the form

$$\Delta_s' = \pm\,\Delta_r A^{s-\binom{n-1}{m-1}}, \tag{15}$$

In the special case of $m = n-1$, (15) is the familiar theorem concerning a minor of the adjoint. The theorems of this paragraph are well known but are not always carefully stated.

Now let the factor determinants in Theorem V be 2-way determinants $A, B, \ldots, E$ and denote a Scott product of them by S, then the theorem says that

$$\operatorname{cpd}_m S = \operatorname{cpd}_m A \quad \operatorname{cpd}_m B \quad \dots \quad \operatorname{cpd}_m E \tag{16}$$

$$= A^{\binom{n-1}{m-1}} \quad B^{\binom{n-1}{m-1}} \quad \dots \quad E^{\binom{n-1}{m-1}} \tag{17}$$

$$= S^{\binom{n-1}{m-1}},$$

that is:

Theorem VI. *The mth compound of a Scott product S of any number of 2-way determinants of the nth order is equal to $S^{\binom{n-1}{m-1}}$.*

Further, applying Theorem V to S and using (14) and (15), we have:

Theorem VII. *Any minor of order r of the mth compound of a Scott product S of 2-way determinants of the nth order is equal to the algebraic complement of the corresponding minor in the introsigned $(n-m)$th compound of S, multiplied by $S^{r-\binom{n-1}{m-1}}$, or*

$$(\operatorname{cpd}_m S)_r = \pm (\operatorname{cpd}_{n-m}{}' S)_s S^{r-\binom{n-1}{m}}; \tag{18}$$

and also

Theorem VIII. *Any minor of order s of the introsigned $(n-m)$th compound of a Scott product S of 2-way determinants is equal to the algebraic complement of the corresponding minor in the mth compound of S, multiplied by $S_s^{\binom{n-1}{m-1}}$, or*

$$(\operatorname{cpd}_{n-m}{}' S)_s = \pm (\operatorname{cpd}_m S)_r S^{s-\binom{n-1}{m-1}}. \tag{19}$$

This last theorem, indeed, amounts simply to saying that in Theorem VII the qualifying word " introsigned " may be prefixed to " mth compound " instead of to " $(n-m)$th compound," if preferred.

To illustrate Theorem VII, let

$$A = \left|a_{\substack{\pm\pm \\ 1\,1 \\ 2\,2 \\ 3\,3 \\ 4\,4}}\right|, \quad C = \left|c_{\substack{\pm\pm \\ 1\,1 \\ 2\,2 \\ 3\,3 \\ 4\,4}}\right|, \quad S = AC = \left|h_{\substack{\pm+\pm \\ 1\,1\,1 \\ 2\,2\,2 \\ 3\,3\,3 \\ 4\,4\,4}}\right| = \begin{matrix} ac. \\ 111 \\ 222 \\ 333 \\ 444 \end{matrix} \tag{20}$$

In the latter notation let us write in full the 2d compound of S.

$ac^*\ ac\ ac^*\ ac\ ac\ ac$ 111 111 111 112 112 113 222 223 224 223 224 224 $ac\ ac\ ac\ ac\ ac\ ac$ 111 111 111 112 112 113 322 323 324 323 324 324 $ac\ ac\ ac\ ac\ ac\ ac$ 111 111 111 112 112 113 422 423 424 423 424 424 $ac^*\ ac\ ac^*\ ac\ ac\ ac$ 211 211 211 212 212 213 322 323 324 323 324 324 $ac\ ac\ ac\ ac\ ac\ ac$ 211 211 211 212 212 213 422 423 424 423 424 424 $ac\ ac\ ac\ ac\ ac\ ac$ 311 311 311 312 312 313 422 423 424 423 424 424	$ac\ ac\ ac\ ac\ ac\ ac$ 111 111 111 112 112 113 232 233 234 233 234 234 $ac\ ac\ ac\ ac\ ac\ ac$ 111 111 111 112 112 113 332 333 334 333 334 334 $ac\ ac\ ac\ ac\ ac\ ac$ 111 111 111 112 112 113 432 433 434 433 434 434 $ac\ ac\ ac\ ac\ ac\ ac$ 211 211 211 212 212 213 332 333 334 333 334 334 $ac\ ac\ ac\ ac\ ac\ ac$ 211 211 211 212 212 213 432 433 434 433 434 434 $ac\ ac\ ac\ ac\ ac\ ac$ 311 311 311 312 312 313 432 433 434 433 434 434	$ac^*\ ac\ ac^*\ ac\ ac\ ac$ 111 111 111 112 112 113 242 243 244 243 244 244 $ac\ ac\ ac\ ac\ ac\ ac$ 111 111 111 112 112 113 342 343 344 343 344 344 $ac\ ac\ ac\ ac\ ac\ ac$ 111 111 111 112 112 113 442 443 444 443 444 444 $ac^*\ ac\ ac^*\ ac\ ac\ ac$ 211 211 211 212 212 213 342 343 344 343 344 344 $ac\ ac\ ac\ ac\ ac\ ac$ 211 211 211 212 212 213 442 443 444 443 444 444 $ac\ ac\ ac\ ac\ ac\ ac$ 311 311 311 312 312 313 442 443 444 443 444 444

$+\ -\ +\ +\ -\ +$ $ac\ ac\ ac\ ac\ ac\ ac$ 333 332 332 331 331 331 444 444 443 444 443 442 $-\ +\ -\ -\ +\ -$ $ac\ ac\ ac\ ac\ ac\ ac$ 233 232 232 231 231 231 444 444 443 444 443 442 $+\ -\ +\ +\ -\ +$ $ac\ ac\ ac\ ac\ ac\ ac$ 233 232 232 231 231 231 344 344 343 344 343 342 $+\ -\ +\ +\ -\ +$ $ac\ ac\ ac\ ac\ ac\ ac$ 133 132 132 131 131 131 444 444 443 444 443 442 $-\ +\ -\ -\ +\ -$ $ac\ ac\ ac\ ac\ ac\ ac$ 133 132 132 131 131 131 344 344 343 344 343 342 $+\ -\ +\ +\ -\ +$ $ac\ ac\ ac\ ac\ ac\ ac$ 133 132 132 131 131 131 244 244 243 244 243 242	$+\ -\ +\ +\ -\ +$ $ac\ ac\ ac\ ac\ ac\ ac$ 323 322 322 321 321 321 444 444 443 444 443 442 $-\ +\ -\ -\ +\ -$ $ac\ ac^*\ ac\ ac^*\ ac^*\ ac^*$ 223 222 222 221 221 221 444 444 443 444 443 442 $+\ -\ +\ +\ -\ +$ $ac\ ac^*\ ac\ ac^*\ ac^*\ ac^*$ 223 222 222 221 221 221 344 344 343 344 343 342 $+\ -\ +\ +\ -\ +$ $ac\ ac\ ac\ ac\ ac\ ac$ 123 122 122 121 121 121 444 444 443 444 443 442 $-\ +\ -\ -\ +\ -$ $ac\ ac^*\ ac\ ac^*\ ac^*\ ac^*$ 123 122 122 121 121 121 344 344 343 344 343 342 $+\ -\ +\ +\ -\ +$ $ac\ ac^*\ ac\ ac^*\ ac^*\ ac^*$ 123 122 122 121 121 121 244 244 243 244 243 242	$+\ -\ +\ +\ -\ +$ $ac\ ac\ ac\ ac\ ac\ ac$ 323 322 322 321 321 321 434 434 433 434 433 432 $-\ +\ -\ -\ +\ -$ $ac\ ac\ ac\ ac\ ac\ ac$ 223 222 222 221 221 221 434 434 433 434 433 432 $+\ -\ +\ +\ -\ +$ $ac\ ac\ ac\ ac\ ac\ ac$ 223 222 222 221 221 221 334 334 333 334 333 332 $+\ -\ +\ +\ -\ +$ $ac\ ac\ ac\ ac\ ac\ ac$ 123 122 122 121 121 121 434 434 433 434 433 432 $-\ +\ -\ -\ +\ -$ $ac\ ac\ ac\ ac\ ac\ ac$ 123 122 122 121 121 121 334 334 333 334 333 332 $+\ -\ +\ +\ -\ +$ $ac\ ac\ ac\ ac\ ac\ ac$ 123 122 122 121 121 121 234 234 233 234 233 232

As $m=2$, and $n=4$, $m-n$ will also equal 2, and therefore the same elements in a different arrangement and with displacement-signs prefixed will appear in the introsigned $(4-2)$th compound, which is also shown in full. Each of the compounds extends across

ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac
121	121	121	122	122	123	121	121	121	122	122	123	131	131	131	132	132	133
232	233	234	233	234	234	242	243	244	243	244	244	242	243	244	243	244	244
ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac
121	121	121	122	122	123	121	121	121	122	122	123	131	131	131	132	132	133
332	333	334	333	334	334	342	343	344	343	344	344	342	343	344	343	344	344
ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac
121	121	121	122	122	123	121	121	121	122	122	123	131	131	131	132	132	133
432	433	434	433	434	434	442	443	444	443	444	444	442	443	444	443	444	444
ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac
221	221	221	222	222	223	221	221	221	222	222	223	231	231	231	232	232	233
332	333	334	333	334	334	342	343	344	343	344	344	342	343	344	343	344	344
ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac
221	221	221	222	222	223	221	221	221	222	222	223	231	231	231	232	232	233
432	433	434	433	434	434	442	443	444	443	444	444	442	443	444	443	444	444
ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac
321	321	321	322	322	323	321	321	321	322	322	323	331	331	331	332	332	333
432	433	434	433	434	434	442	443	444	443	444	444	442	443	444	443	444	444

+	−	+	+	−	+	+	−	+	+	−	+	+	−	+	+	−	+
ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac
313	312	312	311	311	311	313	312	312	311	311	311	313	312	312	311	311	311
444	444	443	444	443	442	434	434	433	434	433	432	424	424	423	424	423	422
−	+	−	−	+	−	−	+	−	−	+	−	−	+	−	−	+	−
ac	ac^*	ac	ac^*	ac^*	ac^*	ac	ac^*	ac	ac^*	ac^*	ac^*	ac	ac^*	ac	ac^*	ac^*	ac^*
213	212	212	211	211	211	213	212	212	211	211	211	213	212	212	211	211	211
444	444	443	444	443	442	434	434	433	434	433	432	424	424	423	424	423	422
+	−	+	+	−	+	+	−	+	+	−	+	+	−	+	+	−	+
ac	ac^*	ac	ac^*	ac^*	ac^*	ac	ac^*	ac	ac^*	ac^*	ac^*	ac	ac^*	ac	ac^*	ac^*	ac^*
213	212	212	211	211	211	213	212	212	211	211	211	213	212	212	211	211	211
344	344	343	344	343	342	334	334	333	334	333	332	324	324	323	324	323	322
+	−	+	+	−	+	+	−	+	+	−	+	+	−	+	+	−	+
ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac	ac
113	112	112	111	111	111	113	112	112	111	111	111	113	112	112	111	111	111
444	444	443	444	443	442	434	434	433	434	433	432	424	424	423	424	423	422
−	+	−	−	+	−	−	+	−	−	+	−	−	+	−	−	+	−
ac	ac^*	ac	ac^*	ac^*	ac^*	ac	ac^*	ac	ac^*	ac^*	ac^*	ac	ac^*	ac	ac^*	ac^*	ac^*
113	112	112	111	111	111	113	112	112	111	111	111	113	112	112	111	111	111
344	344	343	344	343	342	334	334	333	334	333	332	324	324	323	324	323	322
+	−	+	+	−	+	+	−	+	+	−	+	+	−	+	+	−	+
ac	ac^*	ac	ac^*	ac^*	ac^*	ac	ac^*	ac	ac^*	ac^*	ac^*	ac	ac^*	ac	ac^*	ac^*	ac^*
113	112	112	111	111	111	113	112	112	111	111	111	113	112	112	111	111	111
244	244	243	244	243	242	234	234	233	234	233	232	224	224	223	224	223	222

the two pages on which they are printed. In these compounds we have marked with stars the elements of the following minor M of the 2d compound and the elements of the complementary $\overline{M}$ of the corresponding minor of the $(4-2)$th compound.

$$M = \begin{vmatrix} \begin{matrix} ac & ac \\ 111 & 111 \\ 222 & 224 \end{matrix} & \begin{matrix} ac & ac \\ 111 & 111 \\ 242 & 244 \end{matrix} \\ \begin{matrix} ac & ac \\ 211 & 211 \\ 322 & 324 \end{matrix} & \begin{matrix} ac & ac \\ 211 & 211 \\ 342 & 344 \end{matrix} \end{vmatrix}$$

$$= \begin{matrix} ac & ac \\ 111 & 211 \\ 222 & 344 \end{matrix} - \begin{matrix} ac & ac \\ 111 & 211 \\ 224 & 342 \end{matrix} - \begin{matrix} ac & ac \\ 211 & 111 \\ 322 & 244 \end{matrix} + \begin{matrix} ac & ac \\ 211 & 111 \\ 324 & 242 \end{matrix} \tag{21}$$

$$= \begin{matrix} a & c & a & c \\ 11 & 11 & 21 & 11 \\ 22 & 22 & 34 & 44 \end{matrix} - \begin{matrix} a & c & a & c \\ 11 & 11 & 21 & 11 \\ 22 & 24 & 34 & 42 \end{matrix} - \begin{matrix} a & c & a & c \\ 21 & 11 & 11 & 11 \\ 32 & 22 & 24 & 44 \end{matrix} - \begin{matrix} a & c & a & c \\ 21 & 11 & 11 & 11 \\ 32 & 24 & 24 & 42 \end{matrix} \tag{22}$$

$$= (a_{11}a_{22} - a_{21}a_{12})(c_{11}c_{22} - c_{12}c_{21})(a_{21}a_{34} - a_{31}a_{24})(c_{11}c_{44} - c_{14}c_{41}) - \text{etc.} \tag{23}$$

Then we have

$$M \cdot S = -\bar{M} \tag{24}$$

6. **Laws of Complementaries and Extensible Minors.** In the theory of 2-way determinants there are two laws due to Muir,[4] known as the Law of Complementaries and the Law of Extensible Minors, the second resting on the first and this in turn on Jacobi's theorem regarding a minor of the adjoint. Muir also gave a similar theorem regarding a minor of the (not introsigned) $(n-1)$th compound, obtaining from it an alternative Law of Complementaries.[5] From this a somewhat more general Law of Extensible Minors can be obtained. Jacobi's and Muir's theorems are here combined:

Any m-rowed minor of the $\left\{ \begin{matrix} adjoint \\ (n-1)th\ compound \end{matrix} \right\}$ *of a 2-way determinant D of order n is equal to the* $\left\{ \begin{matrix} cofactor \\ complement \end{matrix} \right\}$ *of the corresponding minor of D, multiplied by* D^{n-1}.

The alternative Laws of Complementaries are likewise here combined:

To every general homogeneous relation between a number of the minors of a 2-way determinant there corresponds another relation

[4] T. Muir, *The law of extensible minors in determinants*, Trans. R. Soc. Edinburgh, XXX (1881), pp. 1–4. Abstract in *The Theory of Determinants in the Historical Order of Development*, Sir Thomas Muir, vol. IV, p. 7, Macmillan, 1923. See E. J. Nanson, *On certain determinant theorems*, Crelle's Journ. CXXII. (1900) p. 179.

[5] *Treatise on the Theory of Determinants*, T. Muir, London, Macmillan, 1882, pp. 139, 143.

derivable from the former by merely substituting for every minor its $\left\{\begin{matrix}\textit{cofactor}\\ \textit{complementary}\end{matrix}\right\}$ *in the determinant and then multiplying any term by such a power of the determinant as will make the terms of the same degree.*

The general Law of Complementary Minors may be stated as follows:

To every general homogeneous relation between a number of the minors of a 2-way determinant D of order n there corresponds another relation between minors of any determinant Δ of order $n+h$ of which D is a minor, derivable from the former relation by merely substituting for every minor M of order m that minor of order $m+h$ of Δ which includes M and the complementary $\overline{D}$ of D, and then multiplying any term by such a power of D as will make the terms of the same degree.

For, using the Second Law of Complementaries with respect to D, every minor M is replaced by $\overline{M}.D^{m-1}$. Again using the Law, this time with respect to Δ, $\overline{M}$ (which is of order $n-m$) is replaced by the minor of order $m+h$ which includes M and $\overline{D}$, multiplied by $\Delta^{(n-m)-1}$, while D^{m-1} is replaced by $(\overline{D}.\Delta^{n-1})^{m-1}$. Thus every M is now replaced by an expression containing $\Delta^{m(n-2)}$, and as the original relation was homogeneous Δ completely cancels out, leaving the new relation as stated above.

It is to be noted that D may be any minor of order n of Δ.

The above theory may be extended to Scott products of 2-way determinants. In the previous paper (Theorem 7, p. 62) Jacobi's theorem was thus extended, and similarly Muir's theorem leads to

Theorem XI. *Any minor of order m of the $(n-1)$th compound of a Scott product of 2-way determinants is equal to the complementary of the corresponding minor of the Scott product, multiplied by the $(m-1)$th power of the Scott product.*

The alternative Laws of Complementaries appear in

Theorem XII. *To every homogeneous relation between a number of the minors of a Scott product S of 2-way determinants, the relation being general to that type of Scott product, there corresponds another relation derivable from the former by merely substituting for*

every minor its $\left\{\begin{matrix}\textit{cofactor}\\ \textit{complementary}\end{matrix}\right\}$ *in* S *and then multiplying any term by such a power of* S *as will make the terms of the same degree.*

And the Law of Extensible Minors appears in

Theorem XIII. *To every homogeneous relation between a number of the minors of a Scott product* S *of 2-way determinants of order* n, *the relation being general to that type of Scott product, there corresponds another relation between minors of any like Scott product* T *of order* $n+h$ *of which* S *is a minor, derivable from the former relation by merely substituting for every minor* M *of order* m *that minor of order* $m+h$ *of* T *which includes* M *and the complementary* $\overline{S}$ *of* S, *and then multiplying any term by such a power of* $\overline{S}$ *as will make the terms of the same degree.*

7. **Note on compounds of Cayley product determinants.** In a paper not yet published, we shall show that the mth compound of a Cayley product of two determinants of any classes and any signancies is the like Cayley product of the mth compounds of the factor determinants. As a corollary, the adjoint of the Cayley product is the like Cayley product of the adjoints. But inasmuch as the Cayley product of any number of 2-way determinants is not of higher class, we do not obtain theorems relating to determinants of higher class corresponding to those concerning Scott products in the latter part of the present paper.

INTRODUCTION TO THE RELATIVISTIC INTERPRETATION OF THE QUANTUM THEORY[1]

By TH. DE DONDER[2]

In this paper we shall study electromagnetic systems of one or several degrees of freedom which either occupy a certain volume or are made up of one or several discrete points. We shall define mathematically these systems by means of the following equations:

$$x_i = x_i\,(x_1', x_2', x_3';\ q_1, q_2 \ldots q_f) \quad (i = 1, 2, 3) \tag{1}$$

At a given instant t the system is made up of points (x_1', x_2', x_3') which either fill up a volume or form a discrete distribution of so-called nuclei. f is the number of degrees of freedom. In this system, at the same instant t, the variables $q_1, q_2, \ldots q_f$ have each a constant value. We shall write the latter condition as follows:

$$\delta' q_1 = \delta' q_2 = \ldots \delta' q_f = 0 \tag{2}$$

and

$$\delta' t = 0. \tag{3}$$

If the system in question is continuous, *i.e.*, if it occupies a volume, then we divide this volume into elements which are determined by $\delta' x_1'$, $\delta' x_2'$, $\delta' x_3'$. We have, as before, $\delta' t = 0$. To simplify the exposition of our theory we assume that Equations (1) do not involve t explicitly. We shall then say that the holonomic constraints of the system are independent of the time t. The variable t is introduced only in x_i through the variables $q_1 \ldots q_f$. In other words we have always,

$$\frac{dx_1'}{dt} = \frac{dx_2'}{dt} = \frac{dx_3'}{dt} = 0 \tag{4}$$

and place for the derivatives of $q_1 \ldots q_f$ with respect to t,

$$\dot{q}^1 = \frac{dq_1}{dt}, \ldots \dot{q}^f = \frac{dq_f}{dt}. \tag{5}$$

[1] Paper presented at the Massachusetts Institute of Technology on May 19, 1926, as a part of a course of lectures on "The Mathematical Theory of Relativity."

[2] Professor at the University of Brussels and Special Lecturer at the Massachusetts Institute of Technology for the Spring Term, 1925–1926.

This definition of a system gives us

$$\delta' x_i = \sum_j \frac{\partial x_i}{\partial x_j'} \delta' x_j' \tag{6}$$

and

$$\delta' s^2 = \sum_i \sum_j g_{ij}' \, \delta' x_j' \, \delta' x_i' \qquad (i, j = 1, 2, 3) \tag{7}$$

where we have placed

$$g_{ij}' = \sum_k \sum_l g_{kl} \frac{\partial x_k}{\partial x_i'} \frac{\partial x_l}{\partial x_j'}. \tag{8}$$

For the elementary volumes we shall have

$$\sqrt{-g} \, \delta' x_1 \delta' x_2 \, \delta' x_3 = \sqrt{-g'} \, \delta' x_1' \, \delta' x_2' \delta' x_3'. \tag{9}$$

In the same way,

$$v^i = \frac{dx_i}{dt} = \sum_{\phi=1}^{f} \frac{\partial x_i}{\partial q_\phi} \dot{q}^\phi \qquad (i = 1, 2, 3) \tag{10}$$

To visualize this conception, we consider a rigid body as an example of such a system. Then Equation (1) becomes

$$x_i = x_i^{(0)} + \sum_j x_j' \cos (x_i, x_j') \tag{11}$$

where we immediately recognize the transformation of the rectangular coördinates x_1', x_2', x_3' into the rectangular coördinates x_1, x_2, x_3. The trihedral $(0'; x_1', x_2', x_3')$ is assumed to be attached to the rigid body and thus for each point of this body we shall have the relations (4). Here the parameters q_1, q_2, q_3, are respectively x_1^0, x_2^0, x_3^0, that is, the coördinates of the origin $0'$ with respect to the fixed trihedral $(0; x_1, x_2, x_3)$ at the instant t in question. The parameters q_4, q_5, q_6 are the three Eulerian angles which define the orientation of the trihedral $0'$ with respect to the trihedral 0. We know that the nine cosines which enter in Equation (11) are well-determined functions of q_4, q_5, q_6. If the rigid body is very small and if we pass to the limit, it becomes the point $0'$ for instance. Then we shall have $x_1' = x_2' = x_3' = 0$ for all values of t. We therefore see the advantage of our method, for we still have *six* degrees of freedom. If $0'$ is fixed with respect to the trihedral 0 then we shall say that is a nucleus and we may locate it at the origin 0. Our system has now only *three* degrees of freedom.

In general a molecular system will contain n nuclei if it is possible to choose a system of curvilinear coördinates x_1', x_2', x_3' in such a way that n triplets of *constant* values $x_{1(\nu)}'$, $x_{2(\nu)}'$, $x_{3(\nu)}'$ exist for all values of the time t. The preceding theory may be immediately extended to non-holonomic systems. It is sufficient to write, instead of Equation (10),

$$v^i = \frac{dx_i}{dt} = \sum_{\phi=1}^{f} a_{i\phi}\dot{q}^{\phi} (i = 1, 2, 3) \tag{12}$$

We have shown elsewhere[3] that the following relation holds for the electromagnetic gravific field in space *and* time.

$$\int c^2 \delta m^* \left[\frac{d}{dt}\left(\frac{\partial V}{\partial v^a}\right) - \left(\frac{\partial V}{\partial x_a}\right)\right] + \int \delta e^* \left[\frac{d}{dt}\left(\frac{\partial U^*}{\partial v^a}\right) - \left(\frac{\partial U^*}{\partial x_a}\right)\right] = 0. \tag{13}$$

The significance of the different symbols is explained elsewhere. Applying Equations (1), (2) and (3) we pass from the variables x_i, v^i to the variables q_ϕ and $\dot{q}^\phi$. To begin with we notice that, by (10)

$$\left.\begin{aligned} \left(\frac{\partial V}{\partial q_\phi}\right) &= \sum_i \left(\frac{\partial V}{\partial x_i}\right)\frac{\partial x_i}{\partial q_\phi} + \left(\frac{\partial V}{\partial v^i}\right)\sum_\psi \frac{\partial^2 x_i}{\partial q_\psi \partial q_\phi}\dot{q}^\psi \\ \left(\frac{\partial V}{\partial \dot{q}^\phi}\right) &= \sum_i \left(\frac{\partial V}{\partial v^i}\right)\frac{\partial x_i}{\partial q_\phi} \end{aligned}\right\} \tag{14}$$

From this follows immediately that

$$A_\phi^{(m)} = \sum_{i=1}^{3}\left[\frac{d}{dt}\left(\frac{\partial V}{\partial v^i}\right) - \left(\frac{\partial V}{\partial x_i}\right)\right]\frac{\partial x_i}{\partial q_\phi} \quad (\phi = 1, 2, \ldots f) \tag{15}$$

where we have placed

$$A_\phi^{(m)} = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}}\right) - \left(\frac{\partial V}{\partial q_\phi}\right). \tag{16}$$

Let us recall that V in (16) is a function of q_ϕ and $\dot{q}_\phi$ ($\phi = 1, 2 \ldots f$), that is, by (10)

$$V^2 = \sum_\phi \sum_\psi g_{\phi\psi}^* \dot{q}^\phi \dot{q}^\psi \quad (\phi, \psi = 1, 2, 3) \tag{17}$$

[3] Th. De Donder, "Théorie des champs gravifiques." Mémorial des sciences mathématiques. Paris, Gauthier-Villars, 1926.

or

$$V^2 = \sum_{\phi}\sum_{\psi} g_{\phi}^{\phi\psi}\, \dot{q}_{\phi}\, \dot{q}_{\psi} \,. \tag{18}$$

We have thus

$$g_{\phi\psi}{}^{*} = \sum_{i}\sum_{j} g_{ij} \frac{\partial x_i}{\partial q_{\phi}} \frac{\partial x_j}{\partial q_{\psi}} \,. \tag{19}$$

$$\dot{q}_{\phi} = \sum_{\psi} g_{\phi\psi}{}^{*} \;\; \dot{q}^{\psi} \tag{20}$$

In the same way, bearing in mind Equation (13) we find that

$$A_{\phi}^{(e)} = \sum_{i=1}^{3} \left[\frac{d}{dt}\left(\frac{\partial U^{*}}{\partial v^{i}}\right) - \left(\frac{\partial U^{*}}{\partial x_i}\right)\right] \frac{\partial x_i}{\partial q_{\phi}} \;\; (\phi = 1, 2, \ldots f) \tag{21}$$

where

$$A_{\phi}^{(e)} = \frac{d}{dt}\left(\frac{\partial U^{*}}{\partial \dot{q}^{\phi}}\right) - \left(\frac{\partial U^{*}}{\partial q_{\phi}}\right). \tag{22}$$

Substituting in Equation (13), we obtain, finally

$$\int A_{\phi}^{(m)} c^2 \delta m^{*} + \int A_{\phi}^{(e)} \delta e^{*} = 0. \tag{23}$$

If the system is made up of N electrons or other moving electromagnetic particles, besides n nuclei, (23) becomes, for each electron,

$$A_{\phi\nu}^{(m)} c^2 m_{\nu}{}^{*} + A_{\phi\nu}^{(e)} e_{\nu}{}^{*} = 0 \tag{24}$$

where $m_{\nu}{}^{*}$ and $e_{\nu}{}^{*}$ are the masses and charges of these electrons or particles, respectively. We thus obtain the generalization of the Lagrangian equations by placing

$$L_{\nu}{}^{*} = V_{\nu}{}^{*} c^2 m_{\nu}{}^{*} + U_{\nu}{}^{*} e_{\nu}{}^{*} \tag{25}$$

Let us recall that $U_{\nu}{}^{*}$ is here a function of $q_{\phi\nu}$ and $\dot{q}^{\phi\nu}$ ($\phi = 1, 2 \ldots f$; $\nu = 1, 2 \ldots N$).

In the particular case that there is only one degree of freedom and one electron, *i.e.*, if $f = 1$ and $N = 1$ we shall say that the system is an electromagnetic *oscillator*.

We now come back to the general case and rewrite the foregoing equations using canonical variables. This amounts to an extension to this type of systems of the considerations developed previously (see reference in footnote [3]). Here we place

$$p_\phi^{(m)} = \left(\frac{\partial V}{\partial \dot{q}^\phi}\right), \quad p_\phi^{(e)} = \left(\frac{\partial U^*}{\partial \dot{q}^\phi}\right) \quad (\phi = 1, 2, \ldots f) \tag{26}$$

and introduce the Hamiltonian functions

$$H^{*(m)} = -V + \sum_\phi p_\phi^{(m)} \dot{q}^\phi, \qquad H^{*(e)} = -U^* + \sum_\phi p_\phi^{(e)} \dot{q}^\phi \tag{27}$$

and obtain,

$$\int c^2 \delta m^* \left[\frac{dp_\phi^{(m)}}{dt} + \frac{\delta H^{*(m)}}{\partial q_\phi}\right] + \int \delta e^* \left[\frac{dp_\phi^{(e)}}{dt} + \frac{\delta H^{*(e)}}{\partial q_\phi}\right] = 0. \tag{28}$$

If the system contains N moving particles, Equation (28) becomes, for each particle,

$$c^2 m_\nu^* \left[\frac{dp_{\phi\nu}^{(m)}}{dt} + \left(\frac{\partial H_\nu^{*(m)}}{\partial q_{\phi\nu}}\right)\right] + e_\nu^* \left[\frac{dp_{\phi\nu}^{(e)}}{dt} + \left(\frac{\partial H^{*(e)}}{\partial q_{\phi\nu}}\right)\right] = 0 \tag{29}$$

$$(\nu = 1, 2 \ldots N).$$

Placing

$$H_\nu^* = H_\nu^{*(m)} c^2 m_\nu^* + H_\nu^{*(e)} e_\nu^* \tag{30}$$

we have, by (27) and (29)

$$\frac{dq_{\phi\nu}}{dt} = \frac{\partial H_\nu^*}{\partial p_{\phi\nu}} \quad (\phi = 1, 2 \ldots f),$$

$$\frac{dp_{\phi\nu}}{dt} = -\frac{\partial H_\nu^*}{\partial q_{\phi\nu}} \quad (\nu = 1, 2, \ldots N) \tag{31}$$

where

$$p_{\phi\nu} = c^2 m_\nu^* p_{\phi\nu}^{(m)} + e_\nu^* p_{\phi\nu}^{(e)}. \tag{32}$$

If the N moving particles are all electrons, then all the masses m_ν^* and all the charges e_ν^* are equal and the indices may be removed. In this case (31) become

$$\frac{dq_\phi}{dt} = \frac{\partial H^*}{\partial p_\phi}, \quad \frac{dp_\phi}{dt} = -\frac{\partial H^*}{\partial q_\phi} \quad (\phi = 1, 2 \ldots f) \tag{33}$$

where

$$H^* = H^{*(m)} c^2 m^* + H^{*(e)} e_\nu^*. \tag{34}$$

The preceding considerations have been developed in space and time. They may now be easily generalized to space-time. Equation (1) becomes,

$$x_\alpha = x_\alpha(x_1', x_2', x_3', x_4', ; q_1 \ldots q_f) \quad (\alpha = 1, 2, 3, 4). \tag{35}$$

where x_a represents an event in a given system of reference and x_a' represents the same event in another system of reference such that x_a' *stays constant* when the proper time s is increased by ds. It follows that $dx_a'=0$. Finally $q_1 \ldots q_f$ are functions of s. We introduce the variation δ' such that $\delta's=0$ and $\delta'q=0$, while $\delta'x_1' \ldots \delta'x_4'$ do not vanish.

The conditions (2) and (3) become

$$\delta'q_1=\delta'q_2= \ldots =\delta'q_f=0. \tag{36}$$

$$\delta's=0. \tag{37}$$

The nuclear conditions (4) are now,

$$\frac{dx_1'}{ds}=\frac{dx_2'}{ds}=\frac{dx_3'}{ds}=\frac{dx_4'}{ds}=0. \tag{38}$$

We introduce now the contravariant velocities

$$\kappa^\phi=\frac{dq_\phi}{ds} \quad (\phi=1, 2 \ldots f). \tag{39}$$

From (35) we have

$$\delta's^2=\sum_\alpha\sum_\beta g_{\alpha\beta}'\delta'x_\alpha\delta'x_\beta(\alpha, \beta=1, 2, 3, 4) \tag{40}$$

where we have placed

$$g_{\alpha\beta}'=\sum_a\sum_b g_{ab}\frac{\partial x_a}{\partial x_\alpha'}\frac{\partial x_b}{\partial x_\beta'} \quad (a, b=1, 2, 3, 4). \tag{41}$$

In space-time the elementary volumes are written,

$$\sqrt{-g}\,\delta'x_1\,\delta'x_2\,\delta'x_3\,\delta'x_4=\sqrt{-g'}\,\delta'x_1'\,\delta'x_2'\,\delta'x_3'\,\delta'x_4' \tag{42}$$

where g' is the determinant of the tensor $g_{\alpha\beta}'$ given by (41).

Equation (10) becomes now

$$u^a=\frac{dx_a}{ds}=\sum_{\phi=1}^{f}\frac{\partial x_a}{\partial q_\phi}\kappa^\phi(a=1, 2, 3, 4) \tag{43}$$

By means of the same method whereby (15) was found we shall have here

$$Q_\phi^{(m)}=\sum_{a=1}^{4}\left[\frac{d}{ds}\left(\frac{\partial W}{\partial \kappa^\phi}\right)-\left(\frac{\partial W}{\partial x_a}\right)\right]\frac{\partial x_a}{\partial q_\phi} \tag{44}$$

where

$$Q_\phi^{(m)} = \frac{d}{ds}\left(\frac{\partial W}{\partial \kappa^\phi}\right) - \left(\frac{\partial W}{\partial q_\phi}\right). \tag{45}$$

Let us recall that W in (45) is a function of q_ϕ and κ^ϕ ($\phi = 1, 2, \ldots f$), that is,

$$W^2 = \sum_\phi \sum_\psi g_{\phi\psi}{}^* \kappa^\phi \kappa^\psi \tag{46}$$

or

$$W^2 = \sum_\phi \sum_\psi g^{\phi\psi}{}_* \kappa_\phi \kappa_\psi. \tag{47}$$

We have thus

$$g_{\phi\psi}{}^* = \sum_\alpha \sum_\beta g_{\alpha\beta} \frac{\partial x_\alpha}{\partial q_\phi} \frac{\partial x_\beta}{\partial q_\psi} \tag{48}$$

and

$$\kappa_\phi = \sum_\psi g_{\phi\psi}{}^* \kappa^\psi. \tag{49}$$

We place also,

$$Q_\phi^{(e)} = \sum_{\alpha=1}^{4} \left[\frac{d}{ds}\left(\frac{\partial U}{\partial \kappa^\alpha}\right) - \left(\frac{\partial U}{\partial x_\alpha}\right)\right] \frac{\partial x_\alpha}{\partial q_\phi} \tag{50}$$

where

$$Q_\phi^{(e)} = \frac{d}{ds}\left(\frac{\partial U}{\partial \kappa^\phi}\right) - \left(\frac{\partial U}{\partial q_\phi}\right) \tag{51}$$

and

$$U = \sum_\alpha \phi_\alpha \sum_\phi \frac{\partial x_\alpha}{\partial q_\phi} \kappa^\phi \tag{52}$$

where ϕ_α is the electromagnetic potential. These expressions (50) and (51) satisfy the relation,

$$\int Q_\phi^{(m)} \delta q^{(m)} + \int Q_\phi^{(e)} \delta q^{(e)} = 0 \tag{53}$$

$\delta q^{(m)}$ and $\delta q^{(e)}$ are defined by

$$\delta q^{(e)} = \sigma \delta x_1 \delta x_2 \delta x_3 \delta x_4 \tag{54}$$

$$\delta q^{(m)} = N \delta x_1 \delta x_2 \delta x_3 \delta x_4 \tag{55}$$

σ and N being the electric and mass density factors, respectively.

In the case where the atomic or molecular system reduces to a

number N of electromagnetic particles which can be considered as points, we shall have by (53) and going over to the limit,

$$Q_{\phi\nu}^{(m)} q_\nu^{(m)} + Q_{\phi\nu}^{(e)} q_\nu^{(e)} = 0 \tag{56}$$

where $q_\nu^{(m)}$ and $q_\nu^{(e)}$ are constants characteristic of the νth particle. Equations (56) are in the Lagrangian form; to go over to the canonical form we only have to place

$$H_\nu = -\Lambda_\nu + \sum_\phi p_{\phi\nu} \kappa^\phi \tag{57}$$

where

$$\left.\begin{aligned} \Lambda_\nu &= W q_\nu^{(m)} + U q_\nu^{(e)} \\ p_{\phi\nu} &= \left(\frac{\partial \Lambda_\nu}{\partial \kappa^\phi}\right) \end{aligned}\right\} \tag{58}$$

Equations (56) then become

$$\frac{dq_{\phi\nu}}{ds} = \frac{\partial H_\nu}{\partial p_{\phi\nu}}, \qquad \frac{dp_{\phi\nu}}{ds} = -\frac{\partial H_\nu}{\partial q_{\phi\nu}} \tag{59}$$

with $H_\nu = 0$ ($\nu = 1, 2 \ldots N$). The integration of (59) is thereby reduced to finding a complete integral S_ν of the Hamilton-Jacobi equation,

$$\sum_\psi \sum_\phi g^{\phi\psi} * \left(\frac{\partial S_\nu}{\partial q_\phi} - q_\nu^{(e)} \phi_\phi^*\right)\left(\frac{\partial S_\nu}{\partial q_\psi} - q_\nu^{(e)} \phi_\psi^*\right) = (q_\nu^{(m)})^2 \tag{60}$$

where $g^{\phi\psi}$ is the cofactor of $g_{\phi\psi}^*$ divided by the determinant g^* of $g_{\phi\psi}^*$. The values of x_a' in (59) are those relating to the νth particle.

If identical particles, for example, electrons, are alone considered, then the index ν may evidently be dropped in the above equations, which hold in this case for any one of the particles

We now turn our attention to *statistical mechanics*. We consider a macroscopic system which includes a large number of the systems we have just studied, which we shall call microscopic. To study the former, we introduce the relations,

$$q_\phi = q_\phi(\lambda_1 \ldots \lambda_f;\ \mu_1 \ldots \mu_f), \quad p_\phi = p_\phi(\lambda_1 \ldots \lambda_f;\ \mu_1 \ldots \mu_f) \tag{61}$$

f being the number of degrees of freedom of a microscopic system

and $\lambda_1 \ldots \lambda_f$; $\mu_1 \ldots \mu_f$ two sets of parameters which completely define the system. We have thus an ∞^{2f}-manifold of such systems. It is well known that this manifold is the so-called phase-space of Gibbs. If now we take a certain volume of this phase-space we shall refer to it, following Gibbs, as an extension-in-phase. The latter may be defined by,

$$\int_{2f} \delta q_1 \ldots \delta q_f \, \delta p_1 \ldots \delta p_f \tag{62}$$

or by

$$\int_{(2f)} \frac{\partial(q, p)}{\partial(\lambda, \mu)} \delta\lambda_1 \ldots \delta\lambda_f \delta\mu_1 \ldots \delta\mu_f. \tag{63}$$

We have, on the other hand $\delta s = 0$. From the canonical equations (59) or (33) we know that the extension-in-phase (62) or (63) is an *invariant*, that is

$$\frac{d}{ds}\int_{(2f)} \delta q_1 \ldots \delta q_f \;\; \delta p_1 \ldots \delta p_f = 0 \tag{64}$$

or

$$\frac{d}{dt}\int_{(2f)} \delta q_1 \ldots \delta q_f \, \delta p_1 \ldots \delta p_f = 0. \tag{65}$$

It is also well known that, by (59)

$$\frac{d}{ds}\int \sum_{\phi} p_\phi \, \delta q_\phi = \int \delta\left(- H + \sum_{\phi} p_\phi \frac{\partial H}{\partial p_\phi} \right). \tag{66}$$

Thus, for a *closed* line taken in phase-space, the integral appearing in the left-hand side of (66) is an invariant. In this case we may also write,

$$\frac{d}{ds}\int \sum_{\phi} \delta q_\phi \, \delta p_\phi = 0 \tag{67}$$

where the integral is taken over a two-dimensional manifold bounded by the closed line. The same result is immediately obtained in space and time, that is, with respect to Equations (33). It is very interesting to notice that this extension-in-phase may be connected with the space-time (46), that is with

$$\delta s^2 = \sum_{\phi}\sum_{\psi} g_{\phi\psi}{}^* \;\; \delta q_\phi \delta q_\psi. \tag{68}$$

Let us denote by g^* the determinant ofthe tensor $g_{\phi\psi}{}^*$. Then we may write

$$\int_{(2f)} \delta q_1 \ldots \delta q_f \, \delta p_1 \ldots \delta p_f = \int_{(2f)} \delta V_q \, \delta V_p \qquad (69)$$

where we have placed

$$\delta V_q = \frac{1}{\sqrt{-g^*}} (\delta q_1 \ldots \delta q_f), \quad \delta V_p = \sqrt{-g}\, (\delta p_1 \ldots \delta p_f). \qquad (70)$$

We thus find the generalization to space-time of the volumes used by Gibbs in his study of microcanonic distributions.[4] It is known that this microcanonic distribution is identical with Boltzmann's ergodic distribution.[5] To define such a distribution let us consider the manifold $H = H_0$. We know that this manifold is invariant with respect to Equations (59) or (33). From this follows that

$$C \int_{(2f-1)} \frac{1}{\dfrac{\partial H}{\partial p_\phi}} (\delta q_1 \ldots \delta q_f \, \delta p_1 \ldots \delta p_{\phi-1} \, \delta p_{\phi+1} \ldots \delta p_f) \qquad (71)$$

is also invariant with respect to Equations (59) or (33) if the integral (71) is taken over $a(2f-1)$-manifold upon the invariant manifold $H = H_0$.[6] C denotes a constant which is determined by the condition that the integral (71) taken over the whole manifold $H = H_0$ shall be equal to one. Under this condition (71) defines the ergodic distribution of the phases in the manifold $H = H_0$.

We now come to the definition of quantization. We have seen in (68) that we may consider in the macroscopic system ∞^{2f} phases q_ϕ, $p_\phi (\phi = 1, 2 \ldots f)$. To quantize this system we must introduce $\infty^{(2f-1)}$ conditions in such a way that all the λ's and μ's are functions of only one arbitrary parameter λ. This new parameter λ will be called the *optical* or *spectral* parameter. If λ can only take a discrete series of values we shall say that these values

[4] J. W. Gibbs, "Elementary Principles of Statistical Mechanics," p. 89, 118, New York, 1902.
[5] L. Boltzmann, "Vorlesungen über Gastheorie" Leipzig, 1898, 2. Teil, p. 89.
[6] Th. De Donder, "Sur la répartition ergodique." Bull. Acad. Roy. de Belgique, p. 211–221, 1913.

determine a line spectrum. If on the other hand, in a certain region, λ varies continuously, then we have a continuous spectrum. If the conditions just referred to have an arithmetical character we shall say that the quantization is discontinuous or arithmetical.[7] If then conditions are given by continuous operations (series, integrals, etc.) then we shall say that the quantization is continuous.[8] A mixed character of these conditions is not ruled out. It is not within the power of relativity, at its present state of development, to give any definite answer to this problem, but relativity reveals where the electromagnetic problem and the statistical problem lie. It shows also the conditions of variance or invariance with respect to Equations (33) or (65) and the arbitrary change of variables $q_1 \ldots q_f$, $p_1 \ldots p_f$ with respect to which the problem must be solved in a general way.

[7] Born, "Problems of Atomic Dynamics," p. 123, Cambridge, 1926.
[8] Born and Wiener, this Journal, Vol. 5, p. 84, 1926.